GAMES WITH DISCONTINUOUS PAYOFFS: A STRENGTHENING OF RENY’S EXISTENCE THEOREM

BY ANDREW MCLENNAN, PAULO K. MONTEIRO, AND RABEE TOURKY

We provide a pure Nash equilibrium existence theorem for games with discontinuous payoffs whose hypotheses are in a number of ways weaker than those of the theorem of Reny (1999). In comparison with Reny's argument, our proof is brief. Our result subsumes a prior existence result of Nishimura and Friedman (1981) that is not covered by his theorem. We use the main result to prove the existence of pure Nash equilibrium in a class of finite games in which agents’ pure strategies are subsets of a given set, and in turn use this to prove the existence of stable configurations for games, similar to those used by Schelling (1971, 1972) to study residential segregation, in which agents choose locations.

KEYWORDS: Noncooperative games, discontinuous payoffs, pure Nash equilibrium, existence of equilibrium, better reply security, multiple restrictional security, diagnosable games, residential segregation.

1. INTRODUCTION

Many important and famous games in economics (e.g., the Hotelling location game, Bertrand competition, Cournot competition with fixed costs, and various auction models) have discontinuous payoffs, and consequently do not satisfy the hypotheses of Nash’s existence proof or its infinite dimensional generalizations, but nonetheless have at least one pure Nash equilibrium. Using an argument that is quite ingenious and involved, Reny (1999) established a result that subsumes earlier equilibrium existence results covering many such examples. His theorem’s hypotheses are simple and weak, and in many cases easy to verify. The result has been applied in novel settings many times since then. (See for example Monteiro and Page (2008).) Largely in response to his work, a number of papers on discontinuous games have appeared recently (Carmona (2005, 2009), Bagh and Jofre (2006), Bich (2006, 2009), Monteiro and Page (2007), Barelli and Soza (2009), Tian (2009), Carbonell-Nicolau (2010, 2011), Prokopovych (2010, 2011), de Castro (2011)). Of these, Barelli and Soza (2009)

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deserves special mention because it adopts many techniques from an earlier version of this paper. Briefly, a key idea in Reny (1999) and here is “securing a payoff” at a strategy profile for the other players by playing a pure strategy that insures that payoff when the profile of other players’ strategies is near the given profile. Barelli and Soza showed that results similar to our existence theorem continue to hold if one allows the securing strategy to be a continuous function of the profile of other players’ strategies.

As Reny explained, the main result of Nishimura and Friedman (1981) and results concerning the existence of Cournot equilibria (Szidarovzky and Yakowitz (1977) and the independent (see Roberts and Sonnenschein (1977)) work of McManus (1962, 1964) and Roberts and Sonnenschein (1976)) seemingly have a different character and are not obvious consequences of his result. Here we provide a generalization of Reny’s theorem that easily implies the Nishimura and Friedman result.

The refinements of Reny’s theorem introduced in Section 3 are of two sorts: (a) we weaken the condition of better reply security at a point by allowing several securing strategies to be used; (b) we allow subcorrespondences of the better reply correspondence to be specified by restricting the agents to subsets of their sets of pure strategies. In Section 5 we present an application of (a) and (b). We prove the existence of pure Nash equilibria for diagnosable games, which are finite games in which each agent chooses from a collection of subsets of a given finite set, and of course payoffs are restricted in a certain way. Section 6 uses this result to establish the existence of pure Nash equilibria for a model in which agents choose locations that is similar in spirit to the games that Schelling (1971, 1972) used to study residential segregation. Whereas Schelling used simulation to study the qualitative properties of myopic (in the sense that future relocations are unanticipated) adjustment dynamics, our result demonstrates the existence of stable configurations, which may be viewed as the possible rest points for the sorts of processes he studied.

The remainder of the paper has the following organization. The next section reviews Reny’s theorem and states a result whose hypotheses are less restrictive than Reny’s, but more restrictive than our main result. We also explain how this result implies the Nishimura–Friedman existence theorem. Section 3 states our main result. After a required combinatoric concept has been introduced in Section 4, Section 5 presents the results concerning diagnosable games. Section 6 presents the location model and uses a particular instance of it to demonstrate that other methods of proving the existence of pure Nash equilibria are inapplicable. Section 7 presents the proof of Theorem 3.4. Reny also introduced two concepts, payoff security and reciprocal upper semicontinuity, which together imply the hypotheses of his main result and which are often easy to verify in applications. Section 8 explains the extension of those concepts to our setting. Section 9 contains some concluding remarks and thoughts about possible extensions.
Our system of notation is largely taken from Reny (1999). There is a fixed normal form game

\[ G = (X_1, \ldots, X_N, u_1, \ldots, u_N), \]

where, for each \( i = 1, \ldots, N \), the \( i \)th player’s strategy set \( X_i \) is a nonempty compact convex subset of a topological vector space, and the \( i \)th player’s payoff function \( u_i \) is a function from the set of strategy profiles \( X = \prod_{i=1}^{N} X_i \) to \( \mathbb{R} \).

We adopt the usual notation for “all players other than \( i \).” Let \( X_{\sim i} = \prod_{j \neq i} X_j \).

If \( x \in X \) is given, \( x_{\sim i} \) denotes the projection of \( x \) on \( X_{\sim i} \). For given \( x_{\sim i} \in X_{\sim i} \) (or \( x \in X \)) and \( y_i \in X_i \), we write \((y_i, x_{\sim i})\) for the strategy profile \( z \in X \) satisfying \( z_i = y_i \) and \( z_j = x_j \) for all \( j \neq i \). We endow \( X \) and each \( X_{\sim i} \) with their product topologies. These conventions will also apply to other normal form games introduced later.

We now review Reny’s theorem. A Nash equilibrium of \( G \) is a point \( x^* \in X \) satisfying

\[ u_i(x^*) \geq u_i(x_i, x_{\sim i}) \]

for all \( i \) and all \( x_i \in X_i \). (This would usually be described as a “pure Nash equilibrium,” but we never refer to mixed equilibria, so we omit the qualifier “pure.”) For each player \( i \), let \( B_i : X \times \mathbb{R} \to X_i \) and \( C_i : X \times \mathbb{R} \to X_i \) be the set-valued mappings

\[ B_i(x, \alpha_i) = \{ y_i \in X_i : u_i(y_i, x_{\sim i}) \geq \alpha_i \} \]

and

\[ C_i(x, \alpha_i) = \text{con} B_i(x, \alpha_i). \]

(Here and below \( \text{con} Z \) is the convex hull of the set \( Z \).) Then \( u_i \) is quasiconcave if and only if \( C_i(x, \alpha_i) = B_i(x, \alpha_i) \) for all \( x \in X \) and \( \alpha_i \in \mathbb{R} \). We say that \( G \) is quasiconcave if each \( u_i \) is quasiconcave.

**Definition 2.1:** A player \( i \) can secure a payoff \( \alpha_i \in \mathbb{R} \) on \( Z \subset X \) if there is some \( y_i \in X_i \) such that \( y_i \in B_i(z, \alpha_i) \) for all \( z \in Z \). We say that \( i \) can secure \( \alpha_i \) at \( x \in X \) if she can secure \( \alpha_i \) on some neighborhood of \( x \).

Throughout we assume that each \( u_i \) is bounded. Let \( u = (u_1, \ldots, u_N) : X \to \mathbb{R}^N \). For \( x \in X \), let \( A(x) \) be the set of \( \alpha \in \mathbb{R}^N \) such that \((x, \alpha)\) is in the closure of the graph of \( u \). Since \( u \) is bounded, each \( A(x) \) is compact.

**Definition 2.2:** The game \( G \) is better reply secure at \( x \in X \) if, for any \( \alpha \in A(x) \), there is some player \( i \) and \( \varepsilon > 0 \) such that player \( i \) can secure \( \alpha_i + \varepsilon \) at \( x \). The game \( G \) is better reply secure if it is better reply secure at every strategy profile that is not a Nash equilibrium.

**Theorem 2.3—Reny (1999):** If \( G \) is quasiconcave and better reply secure, then it has a Nash equilibrium.
To better understand Reny’s result in the context of this paper, we reformulate better reply security.

**Definition 2.4:** The game is $B$-secure on $Z \subset X$ if there is $\alpha \in \mathbb{R}^N$ and $\varepsilon > 0$ such that the following conditions hold:

(i) Every player $i$ can secure $\alpha_i + \varepsilon$ on $Z$.

(ii) For each $z \in Z$, there exists some player $i$ with $u_i(z) < \alpha_i - \varepsilon$, that is, $z_i \notin B_i(z, \alpha_i - \varepsilon)$.

The game is $B$-secure at $x \in X$ if it is $B$-secure on some neighborhood of $x$.

**Lemma 2.5:** For each $x \in X$, the game is better reply secure at $x$ if and only if it is $B$-secure at $x$.

**Proof:** First assume that the game is $B$-secure at $x$, so that it is $B$-secure on a neighborhood $U$ of $x$. Let $\alpha$ and $\varepsilon$ be as in the definition. Each $\alpha' \in A(x)$ is the limit of values of $u$ along some sequence or net converging to $x$, so (ii) implies that there is some $i$ with $\alpha'_i \leq \alpha_i - \varepsilon$. By (i), $i$ can secure $\alpha'_i + \varepsilon$ at $x$, which shows that the game is better reply secure at $x$.

Now assume that the game is better reply secure at $x$. Let $\tau$ be a neighborhood base of $x$. For each $i$ and $\alpha'_i$, there is an $\varepsilon > 0$ such that $\alpha'_i + \varepsilon$ can be secured by $i$ at $x$ if and only if $\alpha'_i < \beta_i$, where

$$\beta_i = \sup_{y \in X} \sup_{U \in \tau} \inf_{z \in U} u_i(y_i, z_i).$$

Since the game is better reply secure at $x$, for each $\alpha' \in A(x)$ there is some player $i$ such that $\beta_i > \alpha'_i$, which implies that the inequality $\beta_i > \alpha''_i + \varepsilon$ holds for some $\varepsilon > 0$ and all $\alpha''_i$ in some neighborhood of $\alpha'$. Since $A(x)$ is compact, it is covered by finitely many such neighborhoods, so we may choose $\varepsilon > 0$ such that for any $\alpha' \in A(x)$, there is some $i$ with $\beta_i > \alpha'_i + 2\varepsilon$. Define $\alpha \in \mathbb{R}^N$ by setting

$$\alpha_i = \beta_i - \varepsilon.$$

In view of the definition of $\beta_i$, for each $i$, player $i$ can secure $\alpha_i + \varepsilon$ at $x$, as per (i).

Aiming at a contradiction, suppose (ii) is false, so for each $U \in \tau$, there is some $z_U \in U$ such that $u_i(z_U) \geq \alpha_i - \varepsilon$ for all $i$. Since $\tau$ is a directed set (ordered by reverse inclusion), the boundedness of the image of $u$ implies that there is a convergent subnet, so there is $\alpha' \in A(x)$ such that $\alpha'_i \geq \alpha_i - \varepsilon = \beta_i - 2\varepsilon$ for all $i$, contrary to what we showed above. Q.E.D.

When $G$ is quasiconcave, the following condition is weaker than $B$-security.

**Definition 2.6:** The game is $C$-secure on $Z \subset X$ if there is an $\alpha \in \mathbb{R}^N$ such that the following conditions hold:
(i) Every player $i$ can secure $\alpha_i$ on $Z$.
(ii) For any $z \in Z$, there exists some player $i$ with $z_i \notin C_i(z, \alpha_i)$.
The game is $C$-secure at $x \in X$ if it is $C$-secure on some neighborhood of $x$.

A simple maximization problem illustrates how getting rid of the $\varepsilon$ may matter. Let $N = 1$ and $X_1 = [0, 1]$ with

$$u_1(x_1) = \begin{cases} 
0, & x_1 = 0, \\
(x_1 - \frac{1}{2})^2, & 0 < x_1 \leq 1.
\end{cases}$$

Setting $\alpha_1 = 1/4$, we see that the problem is $C$-secure at each nonmaximizer $x_1$, that is, each $x_1 \in [0, 1)$, but it can easily be checked that it is not $B$-secure at the point $x_1 = 0$, which is not a maximizer, so it is not better reply secure.

In comparison with Definition 2.4, part (ii) of Definition 2.6 replaces $B_i(z, \alpha_i)$ with $C_i(z, \alpha_i)$, which makes it harder to satisfy in general, but not when $G$ is quasiconcave, so the net effect is to make the hypotheses of the next result weaker than those of Theorem 2.3. The hypotheses of Theorem 3.4 will be weaker still.

**Proposition 2.7:** If the game is $C$-secure at each $x \in X$ that is not a Nash equilibrium, then $G$ has a Nash equilibrium.

Nishimura and Friedman (1981) proved the existence of a Nash equilibrium when each $X_i$ is a nonempty, compact, convex subset of a Euclidean space, $u$ is continuous (but not necessarily quasiconcave), and for any $x$ that is not a Nash equilibrium, there is an agent $i$, a coordinate index $k$, and an open neighborhood $U$ of $x$, such that

$$(y^1_{ik} - x^1_{ik})(y^2_{ik} - x^2_{ik}) > 0$$

whenever $x^1, x^2 \in U$ and $y^1_i$ and $y^2_i$ are best responses for $i$ to $x^1$ and $x^2$, respectively. Using compactness, and the continuity of $u_i$, it is not difficult to show that this is equivalent to $(y^1_{ik} - x_{ik})(y^2_{ik} - x_{ik}) > 0$ for any two best responses $y^1_i$ and $y^2_i$ to $x$. A more general condition that does not depend on the coordinate system, or the assumption of finite dimensionality, is that there is a hyperplane that strictly separates $x_i$ from the set of $i$’s best responses to $x$.

We now show that if $G$ satisfies Nishimura and Friedman’s hypotheses, then it is $C$-secure and thus satisfies the hypotheses of Proposition 2.7. Consider an $x \in X$ that is not a Nash equilibrium. For each $i = 1, \ldots, N$, let $\beta_i$ be the utility for $i$ when other agents play their components of $x_{-i}$ and $i$ plays a best response to $x$. Since $u$ is continuous, for any $\varepsilon > 0$, player $i$ can secure $\beta_i - \varepsilon$ at $x$ by playing such a best response. For any neighborhood $V$ of $B_i(x, \beta_i)$, it is the case that $B_i(x, \beta_i - \varepsilon) \subset V$ when $\varepsilon$ is sufficiently small, and in turn it follows
that there is a neighborhood $U$ of $x$ such that $B_i(z, \beta_i - \varepsilon) \subset V$ for all $z \in U$. It follows that if there is a hyperplane strictly separating $x_i$ from $B_i(x, \beta_i)$, then for sufficiently small $\varepsilon > 0$ and a sufficiently small neighborhood $U$ of $x$, this hyperplane also strictly separates $z_i$ and $B_i(z, \beta_i - \varepsilon)$ for all $z \in U$, in which case $z_i \not\in C_i(z, \beta_i)$. Setting $\alpha = (\beta_1 - \varepsilon, \ldots, \beta_N - \varepsilon)$ gives the required property.

3. THE MAIN RESULT

Our main result weakens the hypotheses of Proposition 2.7 in two ways. The first is to allow multiple securing strategies at a point, with each strategy responsible for securing the payoff in response to profiles in some component of a finite closed cover of a neighborhood of the point. The second weakening is that the analyst is allowed to specify restrictions on the strategies that agents can consider in response to a profile. We introduce these in the next definition.

For each player $i$, fix a set-valued mapping $X_i : X \to X_i$, and let $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_N)$. We call $\mathcal{X}$ a restriction operator. For each $i$, define the set-valued mappings $B_i^\mathcal{X} : X \times \mathbb{R} \to X_i$ and $C_i^\mathcal{X} : X \times \mathbb{R} \to X_i$ by setting

$$B_i^\mathcal{X}(x, \alpha_i) = \{y_i \in X_i(x) : u_i(y_i, x_i) \geq \alpha_i\} \quad \text{and} \quad C_i^\mathcal{X}(x, \alpha_i) = \text{con} B_i^\mathcal{X}(x, \alpha_i).$$

**Definition 3.1:** Player $i$ can multiply $\mathcal{X}$-secure a payoff $\alpha_i \in \mathbb{R}$ on a set $Z \subset X$ if $Z$ is covered by finitely many closed (in $Z$) sets $F^1, \ldots, F^J \subset Z$ such that for each $j$ there is some $y^j_i \in X_i$ satisfying $y^j_i \in B_i^\mathcal{X}(z, \alpha_i)$ for all $z \in F^j$. Player $i$ can multiply $\mathcal{X}$-secure $\alpha_i$ at $x \in X$ if she can multiply $\mathcal{X}$-secure $\alpha_i$ on a neighborhood of $x$.

**Definition 3.2:** For $\alpha \in \mathbb{R}^N$ and $Z \subset X$, the game $G$ is multiply $(\mathcal{X}, \alpha)$-secure on $Z \subset X$ if the following conditions hold:

(i) Each player $i$ can multiply $\mathcal{X}$-secure $\alpha_i$ on $Z$.

(ii) For any $z \in Z$, there is some player $i$ for whom $z_i \not\in C_i^\mathcal{X}(z, \alpha_i)$.

The game is multiply $\mathcal{X}$-secure on $Z$ if there is some $\alpha \in \mathbb{R}^N$ such that it is multiply $(\mathcal{X}, \alpha)$-secure on $Z$, and it is multiply $\mathcal{X}$-secure at $x \in X$ if it is multiply $\mathcal{X}$-secure on some neighborhood of $x$.

Following Reny (1999), we do not require that the $X_i$ are Hausdorff spaces. (As Reny acknowledges, this is a mathematical refinement without any known economic applications.) Thus, for $x \in X$, the set $\{x\}$ need not be closed, and we let $[x]$ denote the closure of $\{x\}$.

**Definition 3.3:** The game $G$ is multiply $\mathcal{X}$-secure if it is multiply $\mathcal{X}$-secure at each $x$ such that $[x]$ does not contain a Nash equilibrium. We say that $G$ is multiply restrictionally (MR) secure if there is a restriction operator $\mathcal{X}$ such that it is multiply $\mathcal{X}$-secure.
Our main result is as follows.

**Theorem 3.4:** If $G$ is MR secure, then it has a Nash equilibrium.\(^2\)

The reader may proceed directly to the proof, which is in Section 7.

4. Rectifiable Pairs

In preparation for the games considered in the next section, we introduce a concept related to combinatoric geometry. Let $S$ be a nonempty finite set, let $A$ be a nonempty set of nonempty subsets of $S$, and let $B$ be a set of subsets of $S$ that is closed under intersection.

**Definition 4.1:** A $B$-zellij is a pair $(X, \{Y_b\}_{b \in B})$ in which $X$ is a nonempty compact subset of a topological vector space, each $Y_b$ is a convex subset of $X$, and $Y_b \subset Y_{b'}$ for all $b, b' \in B$ such that $b \subset b'$.

**Definition 4.2:** The pair $(A, B)$ is rectifiable if there is a $B$-zellij $(X, \{Y_b\}_{b \in B})$ with $X$ convex for which there is a continuous function $f : X \to \mathbb{R}^S$ such that the following conditions hold:

(i) $A = \{\arg \max_{s \in S} f_s(x) : x \in X\}$.

(ii) For each $b \in B$, we have $\{a \in A : a \subset b\} = \{\arg \max_{s \in S} f_s(x) : x \in Y_b\}$.

We illustrate this concept with a rich class of rectifiable pairs. Fix an integer $d \geq 1$. A simplex $\sigma$ in $\mathbb{R}^d$ is the convex hull of a finite (possibly empty) affinely independent set of points $V_\sigma \subset \mathbb{R}^d$, the elements of which are called the vertices of the simplex. A face of $\sigma$ is the convex hull of a (possibly empty or nonproper) subset of $V_\sigma$. A (finite) simplicial complex is a finite collection of simplices $\Sigma$ such that the following conditions hold:

(i) $\tau \in \Sigma$ whenever $\sigma \in \Sigma$ and $\tau$ is a face of $\sigma$.

(ii) For any $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of both $\sigma$ and $\sigma'$.

Let $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. If $T$ is a second simplicial complex in $\mathbb{R}^d$, then $\Sigma$ is said to be a subdivision of $T$ if $|T| = |\Sigma|$ and each $\sigma \in \Sigma$ is contained in some $\tau \in T$.

Fix such a $\Sigma$ and $T$, and let $X = |\Sigma| = |T|$ and $S = \bigcup_{\sigma \in \Sigma} V_\sigma$. For each $\tau \in T$, let $W_\tau = \bigcup_{\sigma \subset \tau} V_\sigma = \tau \cap S$, and let

$$A = \{V_\sigma : \emptyset \neq \sigma \in \Sigma\} \quad \text{and} \quad B = \{W_\tau : \tau \in T\}.$$

Note that $B$ is closed under finite intersection: $W_\tau \cap W_{\tau'} = \tau \cap \tau' \cap S = W_{\tau \cap \tau'}$. For $b = W_\tau \in B$, let $Y_b = \tau$. Clearly $(X, \{Y_b : b \in B\})$ is a $B$-zellij.

**Proposition 4.3:** If $X$ is convex, then the pair $(A, B)$ is rectifiable.

\(^2\)The converse holds as well, but for a trivial reason: if $x^*$ is a Nash equilibrium and $X_i(x) = \{x^*_i\}$ for all $i$ and $x$, then $G$ is multiply $X$-secure.
PROOF: For $x \in X$, let $\sigma_x$ be the smallest simplex in $\Sigma$ containing $x$, and let $f(x)$ be the element of $\mathbb{R}^S$ given by the conditions $f_v(x) = 0$ if $v \notin V_{\sigma_x}$, $\sum f_v(x) = 1$, and $\sum f_v(x)v = x$. Note that $f : X \to \mathbb{R}^S$ is continuous. We need to verify (i) and (ii) of Definition 4.2.

For each $x \in X$, we have $\arg \max_{v \in S} f_v(x) = V_{\sigma}$ for some nonempty face $\sigma$ of $\sigma_x$. On the other hand, for each nonempty $\sigma \in \Sigma$, we have $\arg \max_{v \in S} f_v(x_{\sigma}) = V_{\sigma}$, where $x_{\sigma} = \frac{1}{|\sigma|} \sum_{v \in V_{\sigma}} v$. Therefore (i) holds.

To verify (ii), observe that for $b = W_{\tau} \in B$, we have

$$\{a \in \mathcal{A} : a \subset b\} = \{V_{\sigma} : \sigma \subset \tau\} = \left\{\arg \max_{v \in S} f_v(x_{\sigma}) : \sigma \subset \tau\right\}$$

$$= \left\{\arg \max_{v \in S} f_v(x) : x \in \sigma \subset \tau\right\}$$

$$= \left\{\arg \max_{v \in S} f_v(x) : x \in Y_b\right\}.$$

Q.E.D.

For the location model of Section 6, the most important example of realizability is the following corollary.

COROLLARY 4.4: If $\mathcal{A}$ is the set of all nonempty subsets of $S$ and $B = \mathcal{A} \cup \{\emptyset\}$, then $(\mathcal{A}, B)$ is rectifiable.

PROOF: Let $\Sigma = T$ be the set of convex hulls of subsets of the set of standard unit basis vectors $(0, \ldots, 1, \ldots, 0) \in \mathbb{R}^S$, and apply the last result. Q.E.D.

5. DIAGNOSABLE GAMES

We now study a finite game

$$\Gamma = (\mathcal{A}_1, \ldots, \mathcal{A}_N, v_1, \ldots, v_N)$$

in which, for each $i = 1, \ldots, N$, the set $\mathcal{A}_i$ of pure strategies for agent $i$ is a nonempty set of nonempty subsets of a finite set $S_i$, and each $v_i$ is a function from $\mathcal{A} = \prod_{i=1}^N \mathcal{A}_i$ to $\mathbb{R}$.

DEFINITION 5.1: A diagnostic for $\Gamma$ is a pair $(B, d)$ of $N$-tuples $B = (B_1, \ldots, B_N)$ and $d = (d_1, \ldots, d_N)$ such that for each $i$, $(\mathcal{A}_i, B_i)$ is a rectifiable pair and $d_i$ is a function from $\mathcal{A}_{-i}$ to the set of all subsets of $S_i$, and the following conditions hold:

(D1) If $a \in \mathcal{A}$ is not a Nash equilibrium, then there is an $i$ such that $d_i(a_{-i}) \in B_i$ and $a_i$ is not a subset of $d_i(a_{-i})$.

(D2) If $\mathcal{A}'$ is a subset of $\mathcal{A}$ that is completely ordered by componentwise set inclusion (that is, for all $a, a' \in \mathcal{A}'$, either $a_i \subset a'_i$ for all $i$ or $a'_i \subset a_i$ for all $i$), then for each $i$ there is some $a_i \in \mathcal{A}_i$ such that $a_i \subset \bigcap_{a' \in \mathcal{A}'} d_i(a'_{-i})$. 


We say that $\Gamma$ is *diagnosable* if it has a diagnostic.

**Theorem 5.2:** If $\Gamma$ is diagnosable, then it has a Nash equilibrium.

**Proof:** For each $i$, let $E_i = \mathbb{R}^{S_i}$, $S$ be the disjoint union of the sets $S_1, \ldots, S_N$. Let $E = \mathbb{R}^{S} = \prod_{i=1}^{N} E_i$, and consider a strict complete ordering $\succ$ of $S$. For each $i$, let $s_i^*$ be the $\succ$-maximal element of $S_i$. We say that $\beta \in E$ conforms to $\succ$ if $s_i^* \in \arg\max_{s_i \in S_i} \beta_{is_i}$ for all $i$, and

\[
(\ast) \quad \beta_{is_i} - \beta_{js_j} \leq \beta_{js_j} - \beta_{js_i}
\]

for all $i$ and $j$ and all $s_i \in S_i$ and $s_j \in S_j$ such that $s_i \succ s_j$. Let $C_{\succ}$ be the set of all vectors in $E$ that conform to $\succ$. Since $C_{\succ}$ is defined by weak inequalities, it is closed. For any $\beta \in E$, we can define a relation $\succeq$ on $S$ by choosing $s_i^* \in \arg\max_{s_i \in S_i} \beta_{is_i}$ for each $i$ and specifying that for all $i$ and $j$ and all $s_i \in S_i$ and $s_j \in S_j$, $s_i \succeq s_j$ if and only if ($\ast$) holds. Then $\beta \in C_{\succ}$ for any $\succ$ that refines $\succeq$ in the sense that $s_i \succeq s_j$ whenever it is not the case that $s_j \succeq s_i$. Therefore, the various $C_{\succ}$ constitute a closed cover of $E$.

Let $(B, d)$ be a diagnostic for $\Gamma$. For each $i$, let $X_i, Y_{bi} \subset X_i$ for $b_i \in B_i$, and let $f_i: X_i \rightarrow E_i$ be the sets and functions from the rectifiability assumption. Set $X = \prod_{i=1}^{N} X_i$ and define $f: X \rightarrow E$ by setting $f(x) = (f_1(x_1), \ldots, f_N(x_N))$. For each strict complete ordering $\succ$ of $S$, let $F_{\succ} = f^{-1}(C_{\succ})$. Since $f$ is continuous and $C_{\succ}$ is closed, $F_{\succ}$ is closed. Let $\mathcal{F}$ be the collection of nonempty $F_{\succ}$; this is a closed cover of $X$.

For each $i$, let $a_i: X_i \rightarrow A_i$ be the function

\[
a_i(x_i) = \arg\max_{s_i \in S_i} f_{is_i}(x_i)
\]

and define $a: X \rightarrow A$ by setting $a(x) = (a_1(x_1), \ldots, a_N(x_N))$. Fix $\succ$ such that $F_{\succ}$ is nonempty. Let $a^1 = (\{s_1^*, \{s_2^*, \ldots, \{s_N^*\}\})$. For $1 \leq k \leq |S|$, let $i_k$ be the agent such that the $k$th largest (according to $\succ$) element of $S$ is contained in $S_{i_k}$, and let $s_k^i$ be this element. For $k > 1$, define $a^k = (a^1, \ldots, a^k)$ inductively by setting

\[
a^k_i = \begin{cases} a^{k-1} \cup \{s_k^i\}, & \text{if } i_k = i, \\ a^{k-1}, & \text{otherwise.} \end{cases}
\]

If, for $\beta \in C_{\succ}$, $k$ is the largest integer such that $s_k^i \in \arg\max_{s_k \in S_k} \beta_{ik}$, then

\[
a^k = \left(\arg\max_{s_1 \in S_1} \beta_{is_1}, \ldots, \arg\max_{s_N \in S_N} \beta_{Nis_N}\right).
\]

If $z \in F_{\succ}$, then $a(z) = a^k$ for some $k$, so the $a(z)$ for the various $z \in F_{\succ}$ are completely ordered by coordinatewise set inclusion.
We define a game \( G = (X_1, \ldots, X_N, u_1, \ldots, u_N) \) by setting \( u_i(x) = v_i(a(x)) \). For each \( i \), the image of \( a_i \) is \( A_i \) (by (i) of the definition of rectifiability), so \( x^* \in X \) is a Nash equilibrium of \( G \) if and only if \( a(x^*) \) is a Nash equilibrium of \( \Gamma \). We will show that if \( G \) had no Nash equilibria, then it would necessarily be MR secure, after which Theorem 3.4 implies that it has a Nash equilibrium after all, so that \( \Gamma \) also has a Nash equilibrium.

For each \( x \in X \) and player \( i \), let

\[
\mathcal{X}_i(x) = \begin{cases} 
Y_{d_i(a_\cdots(x_{-i}))}, & \text{if } d_i(a_\cdots(x_{-i})) \in B_i, \\
X_i, & \text{otherwise.}
\end{cases}
\]

Note that \( \mathcal{X}_i(x) \) is always convex because \( X_i \) and each \( Y_{b_i} \) are convex. For each \( i \), let \( \alpha_i \) be a lower bound on \( v_i \), so that \( B^\alpha_i(x, \alpha_i) = \mathcal{X}_i(x) = C^\alpha_i(x, \alpha_i) \) for all \( x \in X \). Let \( \alpha = (\alpha_1, \ldots, \alpha_N) \). We need to show that if \( G \) has no Nash equilibria, then \( G \) is multiply \((\mathcal{X}, \alpha)\)-secure at each \( x \in X \), and of course this will follow if we show that \( G \) is multiply \((\mathcal{X}, \alpha)\)-secure on \( X \). This is our goal in the remainder of the proof.

For any \( x \in X \), (D1) tells us that there exists some \( i \) such that \( d_i(a_\cdots(x_{-i})) \in B_i \) and \( a_i(x_i) \) is not a subset of \( d_i(a_\cdots(x_{-i})) \). This implies, by (ii) of Definition 4.2, that \( x_i \notin \mathcal{X}_i(x) = C^\alpha_i(x, \alpha_i) \), which is to say that Definition 3.2(ii) holds for \( Z = X \). It remains to show that Definition 3.2(i) also holds, that is, each \( i \) can multiply \( \mathcal{X} \)-secure \( \alpha_i \) on \( X \). Fixing \( > \) such that \( F_\succ \) is nonempty and fixing a particular \( i \), it suffices to find a \( y^*_i \in \bigcap_{x \in F_\succ} B^\alpha_i(x, \alpha_i) \).

There are now two possibilities. The first is that \( d_i(a_\cdots(x_{-i})) \notin B_i \) for all \( x \in F_\succ \). In this case, \( B^\alpha_i(x, \alpha_i) = \mathcal{X}_i(x) = X_i \) for all \( x \in F_\succ \), so \( y^*_i \) can be any element of \( X_i \).

Otherwise let \( b^*_i = \bigcap d_i(a_\cdots(x_{-i})) \), where the intersection is over all \( x \in F_\succ \) such that \( d_i(a_\cdots(x_{-i})) \in B_i \). We have \( b^*_i \in B_i \) because \( B_i \) is closed under intersection. Since the \( a_\cdots(x_{-i}) \) for the various \( x \in F_\succ \) are completely ordered by coordinatewise set inclusion, (D2) gives an \( a^*_i \in A_i \) such that \( a^*_i \subseteq \bigcap_{x \in F_\succ} d_i(a_\cdots(x_{-i})) \), so that \( a^*_i \subseteq b^*_i \). Condition (ii) of Definition 4.2 now gives a \( y^*_i \in Y_{b^*_i} \), \( y^*_i \notin Y_{b^*_i} \) such that \( a_i(y^*_i) = a^*_i \). For all \( x \in F_\succ \), such that \( d_i(a_\cdots(x_{-i})) \in B_i \), we have \( y^*_i \in Y_{b^*_i} \), and the definition of a zellij gives \( Y_{b^*_i} \subseteq Y_{d_i(a_\cdots(x_{-i}))} \). Therefore, \( y^*_i \in \bigcap_{x \in F_\succ} B^\alpha_i(x, \alpha_i) \).

Q.E.D.

We now show that Theorem 5.2 can be used to give a brief (in comparison with, for example, McLennan and Tourky (2008)) proof of Sperner’s lemma. This is significant because it shows that Theorem 5.2 is not simpler than the fixed point principle and, consequently, not likely to be provable by more elementary methods.

Let \( E = \{ e_1, \ldots, e_d \} \) be the set of standard unit basis vectors of \( \mathbb{R}^d \). Let \( \Sigma \) be a simplicial complex such that \( |\Sigma| \) is the convex hull of \( E \), and let \( V = \bigcup_{e \in \Sigma} V_e \).

For each \( F \subset E \), let \( W_F \) be the set of \( v \in V \) contained in the convex hull of \( F \).

A Sperner labelling is a function \( \lambda : V \to E \) such that \( \lambda(W_F) \subset F \) for all \( F \subset E \).
A simplex $\sigma \in \Sigma$ is completely labelled if $\lambda(V_{\sigma}) = E$. Sperner’s lemma asserts that any Sperner labelling has a completely labelled simplex.

We now define a two player game $\Gamma = (A_1, A_2, v_1, v_2)$ in which $A_1$ is the set of all nonempty subsets of $E$ and $A_2 = \{V_{\sigma} : \emptyset \neq \sigma \in \Sigma\}$. Note that $A_2$ contains each singleton subset of $E$. Let $r : E \to E$ be the function given by $r(e_i) = e_i + 1$ if $i < d$ and $r(e_d) = e_1$. The payoffs of the players are given by

$$v_1(a_1, a_2) = \begin{cases} 1, & \text{if } a_1 \subseteq r(\lambda(a_2)), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$v_2(a_1, a_2) = \begin{cases} 1, & \text{if } a_2 \subseteq W_{a_1}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $B_1 = A_1 \cup \{\emptyset\}$ and $B_2 = \{W_F : F \subset E\}$. Proposition 4.3 implies that $(A_1, B_1)$ and $(A_2, B_2)$ are rectifiable. Let $d_1(a_2) = r(a_2)$ and $d_2(a_1) = W_{a_1}$. If $(a_1, a_2)$ is not a Nash equilibrium, then either $a_1 \nsubseteq r(a_2) = d_1(a_2) \in B_1$ or $a_2 \nsubseteq W_{a_1} = d_2(a_1) \in B_2$. Therefore, (D1) is satisfied. For (D2), note that $d_1(a_2)$ and $d_2(a_1)$ are strictly monotonic (with respect to set inclusion) functions of $a_2$ and $a_1$, and are never empty. Thus $\Gamma$ is a game on subsets, so it has a Nash equilibrium $a^*$.

Each player can obtain utility 1 by responding appropriately to a given pure strategy of the other player: $v_1(a_1, a_2) = 1$ if $a_1 = r(\lambda(a_2))$ and $v_2(a_1, a_2) = 1$ if $a_2$ is a singleton subset of $a_1$. Therefore, $v_1(a_1^*, a_2^*) = v_2(a_1^*, a_2^*) = 1$, which is to say that $a_1^* \subseteq r(\lambda(a_2^*))$ and $a_2^* \subseteq W_{a_1^*}$. Since $\lambda$ is a Sperner labelling, the latter condition implies that $\lambda(a_2^*) \subseteq a_1^*$, so $\lambda(a_2^*) \subset r(\lambda(a_2^*))$. If $\emptyset \neq F \subset E$ and $F \subset r(F)$, then $F = E$, so $\lambda(a_2^*) = E$. That is, the $\sigma$ such that $a_2^* = V_{\sigma}$ is completely labelled.

### 6. A LOCATION MODEL

Schelling (1971, 1972) explored a number of dynamic processes involving location choices by individuals, emphasizing the relationship between individual preferences to be co-located with similar people and segregated outcomes. More generally, the idea of locational externalities goes back at least to Marshall, and there is now a large related literature that continues to be quite active.

For the most part, Schelling did not work with configurations that are stable, in the sense that each agent prefers her current location to any available alternative, taking the locations of others as fixed, and did not prove that such configurations exist. Instead, he presented simulations in which agents repeatedly relocate to preferred points myopically, insofar as agents do not anticipate the future relocation decisions of others.

This section presents finite games that are similar in spirit, but instead of studying them dynamically, we show that such a game has a nonempty set of
Nash equilibria, which can be interpreted as stable configurations of location choices. This means that the associated dynamic model has configurations that are potential rest points and which in this sense can be regarded as potential ultimate outcomes.

**Definition 6.1:** A location game is a game \( \Gamma = (A_1, \ldots, A_N, v_1, \ldots, v_N) \) such that, for some nonempty finite set \( S \) of locations, for each \( i \), it is the case that the following statements hold:

(i) \( A_i \) is the set of all nonempty subsets of some nonempty \( S_i \subset S \).

(ii) There are sets \( U_i, D_i \subset \{1, \ldots, N\} \setminus \{i\} \). For \( s \in S_i \) and \( a_{-i} \in A_{-i} \) let

\[
u_i(s, a_{-i}) = |\{j \in U_i : s \in a_j\}| \quad \text{and} \quad d_i(s, a_{-i}) = |\{j \in D_i : a_j = \{s\}\}|.
\]

(iii) The payoff of player \( i \) at strategy profile \( a \in A \) is \( v_i(a) = \min_{s \in a_i} y_i(s, a_{-i}) \), where two conditions hold:

(a) If \( U_i \neq \emptyset \), then \( y_i : S_i \times A_{-i} \to \{0, 1\} \) is given by

\[
y_i(s, a_{-i}) = \begin{cases} 
0, & \text{if } u_i(s, a_{-i}) > d_i(s, a_{-i}), \\
1, & \text{otherwise}.
\end{cases}
\]

(b) If \( U_i = \emptyset \), then \( y_i : S_i \times A_{-i} \to \{0, 1\} \) is given by

\[
y_i(s, a_{-i}) = \begin{cases} 
0, & \text{if } s \notin \bigcup_{j \in D_i} a_j, \\
1, & \text{otherwise}.
\end{cases}
\]

We think of \( S_i \) as the set of locations that are feasible for agent \( i \). Perhaps these are bars close to where \( i \) lives. The strategy \( a_i \) is the set of locations that \( i \) sometimes visits. We say that player \( i \) frequents each \( s \in a_i \) and that \( i \) is a denizen of \( s \) if \( a_i = \{s\} \).

The sets \( U_i \) and \( D_i \) are the sets of other people that agent \( i \) regards as undesirable and desirable, respectively. (Note that it is not necessary to assume that \( U_i \) and \( D_i \) are disjoint, although that would be consistent with the spirit of the model.) Thus \( u_i(s, a_{-i}) \) and \( d_i(s, a_{-i}) \) are, respectively, the number of undesirables frequenting \( s \) and the number of desirable denizens of \( s \) from \( i \)'s point of view. If \( U_i \) is nonempty, then player \( i \)'s only concern is to avoid frequenting a location where the undesirables might outnumber the desirables because it is frequented by more undesirables than the number of desirable denizens. If \( i \) does not regard anyone as undesirable, then she only wishes to avoid frequenting a location that is not frequented by anyone she likes.

**Theorem 6.2:** A location game \( \Gamma = (A_1, \ldots, A_N, v_1, \ldots, v_N) \) is diagnosable and, consequently, has a Nash equilibrium.
PROOF: For each i, let $\mathcal{B}_i = \mathcal{A}_i \cup \{\emptyset\}$. The pair $(\mathcal{A}_i, \mathcal{B}_i)$ is rectifiable by Corollary 4.4. Let $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$. For each i, define $d_i$ by setting $d_i(a_{-i}) = \arg \max_{s \in S_i} y_i(s, a_{-i})$. We need to verify (D1) and (D2).

If $a \in \mathcal{A}$ is not a Nash equilibrium, then for some i, we have $v_i(a) = 0$ and $v_i(a', a_{-i}) = 1$ for some $a' \in \mathcal{A}_i$, so that $y_i(s, a_{-i}) = 0$ for some $s \in a_i$ and $y_i(s', a_{-i}) = 1$ for all $s' \in a'_i$. Thus $s \notin d_i(a_{-i})$, so $a_i$ is not subset of $d_i(a_{-i})$. Since any subset of $S_i$ is an element of $\mathcal{B}_i$, it follows that (D1) holds.

Let $a^1, a^2, \ldots, a^K$ be a sequence in $\mathcal{A}$ that is decreasing with respect to componentwise set inclusion. Fix a player i and note that for any s, $u_i(s, a^k_i)$ is a weakly decreasing function of k and $d_i(s, a^k_i)$ is a weakly increasing function of k.

First suppose that $U_i$ is not empty. If $u_i(s, a^k_i) > d_i(s, a^k_i)$ for all $s \in S_i$ and all $k$, then for any $s^*$, we have $s^* \in d_i(a^{k_1}_i) \cap \cdots \cap d_i(a^{k_n}_i)$. Otherwise, let $k^*$ be the first number such that $u_i(s^*, a^{k_1}_i) \leq d_i(s^*, a^{k_1}_i)$ for some $s^* \in S_i$. Then $u_i(s^*, a^{k_1}_i) \leq d_i(s^{k_1}_i, a^{k_1}_i)$ and, consequently, $y_i(s^{k_1}_i, a^{k_1}_i) = 1$ for all $k \geq k^*$. If $k < k^*$, then $u_i(s, a^{k_1}_i) > d_i(s, a^{k_1}_i)$ and $y_i(s, a^{k_1}_i) = 0$ for all $s \in S_i$, so that $s \notin d_i(a^{k_1}_i)$ for all $s \in S_i$. In particular, for all $k$, we have $s^* \in d_i(a^{k_1}_i)$. Letting $a_i = \{s^*\}$, we conclude that (D2) is satisfied.

Suppose now that $U_i$ is empty. If $a^{j}_i \cap S_i = \emptyset$ for all $j \in D_i$, then for any $s^* \in S_i$, we have $s^* \in d_i(a^{k_1}_i)$ for all $k$. Otherwise, let $k^*$ be the largest integer such that $a^{k_1}_j \cap S_i \neq \emptyset$ for some $j \in D_i$ and let $s^*$ be an element of such an intersection. Again we have $s^* \in d_i(a^{k_1}_i)$ for all $k$. We conclude that (D2) is also satisfied in this case.

Q.E.D.

Insofar as we wish to convincingly illustrate the usefulness of our results, it is important to consider whether other methods could be used to show that a game on subsets has a Nash equilibrium. The following particular instance of the location model illustrates how various other methods are inapplicable.

There are two players and three locations. Let $S = \{a, b, c\}$, $S_1 = S$, and $S_2 = \{a, b\}$. Player 1 considers player 2 desirable while player 2 considers player 1 undesirable, so that $U_1 = \emptyset$, $D_1 = \{2\}$, $U_2 = \{1\}$, and $D_2 = \emptyset$. Therefore, the payoffs are

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{a}</td>
</tr>
<tr>
<td></td>
<td>(1,0)</td>
</tr>
<tr>
<td>{b}</td>
<td>(0,1)</td>
</tr>
<tr>
<td>{c}</td>
<td>(0,1)</td>
</tr>
<tr>
<td>{a, b}</td>
<td>(0,0)</td>
</tr>
<tr>
<td>{a, c}</td>
<td>(0,0)</td>
</tr>
<tr>
<td>{b, c}</td>
<td>(0,1)</td>
</tr>
<tr>
<td>{a, b, c}</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>
One can easily check that \( (\{a, b\}, \{a, b\}) \) is the only pure strategy Nash equilibrium.

There is a mixed Nash equilibrium \( (\frac{1}{2}\{a\} + \frac{1}{2}\{b\}, \frac{1}{2}\{a\} + \frac{1}{2}\{b\}) \) that is not a pure equilibrium. Therefore, one cannot prove the existence of a pure Nash equilibrium by combining the existence of a mixed equilibrium with an argument showing that all equilibria are pure.

A strategic form game is a potential game (Rosenthal (1973), Monderer and Shapley (1996)) if there is a real-valued potential function on the set of pure strategy profiles such that for any pair of profiles that differ in only one agent’s component, the difference in that agent’s utility at the two profiles has the same sign as the difference in the potential function at the two profiles. Any maximizer of the potential function is a Nash equilibrium, so Nash equilibria exist when the pure strategy sets are finite. For any potential game, any game obtained by restricting each agent to a subset of her strategies is also a potential game. The game obtained by restricting each agent to play either \( \{a\} \) or \( \{b\} \) is an instance of matching pennies, which of course has no Nash equilibrium and is thus not a potential game.

A strategic form game exhibits strategic complementarities or increasing differences (Bulow, Geanakoplos, and Klemperer (1985)) if the agents’ sets of pure strategies are partially ordered and increasing one agent’s strategy, relative to the given ordering, weakly increases the desirability for the other agents to increase their strategies. If various auxiliary conditions are satisfied, a game that exhibits strategic complementarities necessarily has Nash equilibria. In a game on subsets, the pure strategies are ordered by both inclusion and reverse inclusion. However, we have

\[
v_1(\{a, b\}, \{a, b\}) - v_1(\{a\}, \{a\}) > v_1(\{a, b\}, \{a\}) - v_1(\{a\}, \{a\})
\]

and

\[
v_1(\{a, b, c\}, \{a, b\}) - v_1(\{a, b\}, \{a, b\}) < v_1(\{a, b, c\}, \{a\}) - v_1(\{a, b\}, \{a\}),
\]

so the location game does not exhibit strategic complementarities.

This example can also be used to show that Reny’s theorem cannot be used to prove Theorem 6.2 by following the proof of Theorem 5.2 up to the construction of the derived game \( G \), then showing that \( G \) is better reply secure when \( I \) is a location game. In this example, \( A_1 \) is the set of all nonempty subsets of \( S_1 \), and \( B_1 = A_1 \cup \{\emptyset\} \). As Corollary 4.4 points out, \( (A_1, B_1) \) is rectifiable because we can follow the proof of Proposition 4.3 with \( \Sigma_1 = T_1 \) being the set of all faces of the simplex \( \Delta_1 = \{x_1 \in \mathbb{R}_{+}^{S_1} : x_{1a} + x_{1b} + x_{1c} = 1\} \), setting \( X_1 = |\Sigma_1| = |T_1| = \Delta_1 \) and letting \( f_1 : X_1 \rightarrow \mathbb{R}^{S_1} \) be the identity. Similarly, \( \Sigma_2 = T_2 \) is the set of all faces of the simplex \( \Delta_2 = \{x_2 \in \mathbb{R}_{+}^{S_2} : x_{2a} + x_{2b} = 1\} \), \( X_2 = |\Sigma_2| = |T_2| = \Delta_2 \), and
$f_2 : X_2 \to \mathbb{R}^{S_2}$ is the identity. The proof of Theorem 5.2 constructs the game $G = (X_1, X_2; u_1, u_2)$ by setting

$$u_i(x) = v_i \left( \arg \max_{s_1 \in S_1} x_{1s_1}, \arg \max_{s_2 \in S_2} x_{2s_2} \right)$$

for both $i$. The point $((0, 0, 1), (1/2, 1/2))$ is not a Nash equilibrium, and it is not hard to show that the game is not better reply secure at this point, so $G$ is not better reply secure.

It can happen that a game has an upper hemicontinuous best reply correspondence even though the payoff functions are discontinuous, but one can see that this is not the case for $G$ by considering a sequence $x^n_1$ which correspond to player 1 playing $\{a\}$ converging to $x_1$ that corresponds to $\{a, b\}$. Note that player 2’s best response to player 1 playing $x_1$ does not include $(1, 0)$. Therefore, the best response correspondence is not upper hemicontinuous.

7. THE PROOF OF THEOREM 3.4

We say that a (not necessarily Hausdorff) topological space is regular if each point has a neighborhood base of closed sets. (This is the definition given by Kelley (1955), which differs from the usage of some other authors.) Topological vector spaces are regular topological spaces, even if they are not Hausdorff (e.g., Schaefer (1971, p. 16)). It is easy to see that any subspace of a regular space is regular and that finite cartesian products of regular spaces are regular, so each $X_i$, each $X_{-i}$, and $X$ are all regular.

In preparation for the main body of the argument, we present two lemmas, the first of which is a variant of the fixed point principle that holds in topological vector spaces that are neither Hausdorff nor locally convex. Although it bears some resemblance to the Knaster–Kuratowski–Mazurkiewicz lemma, we have not managed to forge a direct connection between the two results.

**Lemma 7.1:** Let $X$ be a nonempty convex subset of a topological vector space and let $P : X \to X$ be a set-valued mapping. If there is a finite closed cover $H_1, \ldots, H_L$ of $X$ such that $\bigcap_{z \in H_j} P(z) \neq \emptyset$ for each $j = 1, \ldots, L$, then there exists $x^* \in X$ such that $x^* \in \text{con} P(x^*)$.

**Proof:** For each $j = 1, \ldots, L$, choose a $y_j \in \bigcap_{z \in H_j} P(z)$. Let $e_1, \ldots, e_L$ be the standard unit basis vectors of $\mathbb{R}^L$, let $\Delta$ be their convex hull, and let $\pi : \Delta \to X$ be the map $\omega \mapsto \sum_j \omega_j y_j$; this is continuous because addition and scalar multiplication are continuous operations in any topological vector space. Define a set-valued mapping $Q : \Delta \to \Delta$ by letting $Q(\omega)$ be the convex hull of $\{e_j : \pi(\omega) \in H_j\}$. This is nonempty-valued because the $H_j$ cover $X$, it is upper hemicontinuous because each $H_j$ is closed, and of course it is convex-valued,
so Kakutani’s fixed point theorem implies that it has a fixed point \( \omega^* \). Let \( x^* = \pi(\omega^*) \). Then

\[
x^* \in \text{con}\{ y_j : x^* \in H_j \} \subset \text{con}\left( \bigcup_{j : x^* \in H_j} \left( \bigcap_{x \in H_j} P(x) \right) \right) \subset \text{con} P(x^*).
\]

Q.E.D.

For the remainder of the section we fix a game \( G = (X_1, \ldots, X_N; u_1, \ldots, u_N) \) and a restriction operator \( \mathcal{X} \).

**Lemma 7.2:** Suppose that \( \alpha_1, \ldots, \alpha_\ell \in \mathbb{R}^N \), \( U_1, \ldots, U_\ell \) are subsets of \( X \), and, for each \( h = 1, \ldots, \ell \), \( G \) is multiply \((\mathcal{X}, \alpha_h)\)-secure on \( U_h \). Let

\[
\alpha = \max_{h=1,\ldots,\ell} \alpha^h = \left( \max_{h=1} \alpha^h_1, \ldots, \max_{h=1} \alpha^h_N \right).
\]

Then \( G \) is multiply \((\mathcal{X}, \alpha)\)-secure on any nonempty \( U \subset \bigcap_{h=1}^\ell U_h \).

**Proof:** For each \( h \), there is a cover \( F^1_h, \ldots, F^\ell_h \) of \( U_h \) by relatively closed subsets such that for each \( i \) and \( j \), we have \( \bigcap_{x \in F^j_h} B^X_i(z, \alpha^j_h) \neq \emptyset \). Let \( G^1, \ldots, G^\ell \) be the nonempty intersections of the form \( F^1_h \cap \cdots \cap F^\ell_h \cap U \); this is a cover of \( U \) by nonempty relatively closed subsets. It refines each cover \( F^1_h, \ldots, F^\ell_h \), so for all \( i \) and \( j \), we have \( \bigcap_{x \in G^j} B^X_i(z, \alpha_i) \neq \emptyset \) because there is some \( h \) such that \( \alpha_i = \alpha^h_i \). For any \( h \) and \( z \in U \), there is some \( i \) such that \( z_i \notin C^X_i(z, \alpha^h_i) \). Since \( \alpha_i \geq \alpha^h_i \), this implies that \( z_i \notin C^X_i(z, \alpha_i) \). Q.E.D.

We now have the tools we need to complete the proof of Theorem 3.4. Aiming at a contradiction, suppose that \( G \) is multiply \( \mathcal{X} \)-secure and has no Nash equilibrium. Because \( X \) is compact and regular, it is covered by closed sets \( F_1, \ldots, F_\ell \) such that for each \( h = 1, \ldots, \ell \), there is \( \alpha_h \in \mathbb{R}^N \) and an open neighborhood \( U_h \) of \( F_h \) such that \( G \) is multiply \((\mathcal{X}, \alpha_h)\)-secure on \( U_h \). Define \( \psi : X \to \mathbb{R} \) by setting

\[
\psi(x) = \max_{x \in F_h} \alpha_h.
\]

For each \( x \), let \( U_x = \bigcap_{h : x \in F_h} U_h \). The last result implies that \( G \) is multiply \((\mathcal{X}, \psi(x))\)-secure on any nonempty subset of \( U_x \), whence it is multiply \((\mathcal{X}, \psi(x))\)-secure at \( x \).

For each \( i \) and \( x \in X \), let \( P_i(x) = B^X_i(x, \psi_i(x)) \) and let \( P(x) = P_1(x) \times \cdots \times P_N(x) \). Below we will show that the hypotheses of Lemma 7.1 are satisfied, so some \( x^* \) is an element of \( \text{con} P(x^*) \), which is to say that \( x^*_i \in C^X_i(x^*, \psi_i(x^*)) \) for all \( i \). But \( G \) is multiply \((\mathcal{X}, \psi(x^*))\)-secure at \( x^* \), so \( x^*_i \notin C^X_i(x^*, \psi_i(x^*)) \) for some \( i \), which is the desired contradiction.
Consider a particular $x \in X$. The function $\psi$ is upper semicontinuous and takes on finitely many values, so there is a closed neighborhood $Z_x \subset U_x$ of $x$ such that $\psi(z) \leq \psi(x)$ for all $i$ and $z \in Z_v$. Since $G$ is multiply $(X, \psi(x))$-secure on $Z_x$, there is a cover $Z_1^x, \ldots, Z_{\varepsilon}^x$ of $Z_x$ by closed subsets such that for each $j = 1, \ldots, J$, we have

$$
\bigcap_{z \in Z_j^x} P_i(z) = \bigcap_{z \in Z_j^x} B_i^X(z, \psi_i(z)) \supset \bigcap_{z \in Z_j^x} B_i^X(z, \psi_i(x)) \neq \emptyset
$$

for all $i$, so that

$$
\bigcap_{z \in Z_x^i} P(z) = \bigcap_{z \in Z_x^i} P_1(z) \times \cdots \times P_N(z) = \bigcap_{z \in Z_x^i} P_1(z) \times \cdots \times \bigcap_{z \in Z_x^i} P_N(z) \neq \emptyset.
$$

Since $X$ is compact, it is covered by some finite collection $Z_1^x, \ldots, Z_m^x$, and the various $Z_x^i$ constitute a cover of the sort required by Lemma 7.1.

8. PAYOFF SECURE GAMES

Reny pointed out that a combination of two conditions, payoff security and reciprocal upper semicontinuity, imply the hypotheses of his existence result, and in applications it is typically relatively easy to verify them when they hold. We now define generalizations of these notions in our setting and establish that together they imply that the game is MR secure.

Reny’s notion of payoff security requires that for each $x$ and $\varepsilon > 0$, each player $i$ can multiply $u_i(x) - \varepsilon$ at $x$.

**DEFINITION 8.1:** The game $G$ is multiply $X$-payoff secure at $x \in X$ if, for each $\varepsilon > 0$, each player $i$ can multiply $X$-secure $u_i(x) - \varepsilon$ at $x$. The game $G$ is multiply $X$-payoff secure if it is $X$-secure at each $x \in X$.

Reny’s reciprocal upper semicontinuity (which was introduced by Simon (1987) under the name “complementary discontinuities” and which generalizes the requirement in Dasgupta and Maskin (1986) that the sum of payoffs be upper semicontinuous) requires that $u(x) = \alpha$ whenever $(x, \alpha)$ is in the closure of the graph of $u$ and $\alpha \geq u(x)$. Equivalently, for each possible “jump up” $\delta > 0$, there is a neighborhood $U$ of $x$ and a “jump down” $\varepsilon > 0$ such that for each $z \in U$, if $u_i(z) > u_i(x) + \delta$ for some $i$, then there must be a player $j$ with $u_j(z) < u_j(x) - \varepsilon$.

**DEFINITION 8.2:** The game $G$ is $X$-reciprocally upper semicontinuous at $x \in X$ if for every $\delta > 0$, there exists $\varepsilon > 0$ and a neighborhood $U$ of $x$ such that
for each \( z \in U \), if there is a player \( i \) with \( z_i \in C_i^X(z, u_i(x) + \delta) \), then there is a player \( j \) such that \( z_j \notin C_j^X(z, u_j(x) - \epsilon) \). The game \( G \) is \( X \)-reciprocally upper semicontinuous if it is \( X \)-reciprocally upper semicontinuous at each \( x \in X \).

The next result is the analog in our setting of Reny’s Proposition 3.2, which asserts that payoff security and reciprocal upper semicontinuity imply better reply security. In conjunction with Theorem 3.4, it implies a generalization of Reny’s Corollary 3.3. We say that \( X \) is other-directed if, for each \( i \), \( X_i(x) \) depends only on \( x_{-i} \), so that \( X_i(y_i, x_{-i}) = X_i(z_i, x_{-i}) \) for all \( x_{-i} \in X_{-i} \) and \( y_i, z_i \in X_i \).

**Proposition 8.3:** If \( X \) is other-directed and \( G \) is multiply \( X \)-payoff secure and \( X \)-reciprocally upper semicontinuous, then it is multiply \( X \)-secure.

**Proof:** Fixing an \( x \in X \) that is not a Nash equilibrium, our goal is to show that the game is multiply \( X \)-secure at \( x \). Let \( \mathcal{I} \) be the set of players \( i \) such that \( x_i \) is not a best response to \( x_{-i} \). For each \( i \in \mathcal{I} \), choose \( y_i \) such that \( u_i(y_i, x_{-i}) > u_i(x) \), and let \( \delta > 0 \) be small enough that \( u_i(y_i, x_{-i}) > u_i(x) + \delta \) for all \( i \in \mathcal{I} \). Since the game is multiply \( X \)-payoff secure, each \( i \in \mathcal{I} \) can multiply \( X \)-secure \( \alpha_i = u_i(x) + \delta \) at \( (y_i, x_{-i}) \). Since \( X \) is other-directed, player \( i \) can also multiply \( X \)-secure \( \alpha_i \) at \( x \).

Since the game is \( X \)-reciprocally upper semicontinuous, there is an \( \epsilon > 0 \) and a neighborhood \( U \) of \( x \) such that for all \( z \in U \), if \( z_i \in C_i^X(z, u_i(x) + \delta) \) for some \( i \), then \( z_j \notin C_j^X(z, u_j(x) - \epsilon) \) for some \( j \). Since the game is multiply \( X \)-payoff secure, each \( j \notin \mathcal{I} \) can multiply \( X \)-secure \( \alpha_j = u_j(x) - \epsilon \) at \( x \). It is now the case that for each \( z \in U \), either \( z_i \notin C_i^X(z, \alpha_i) \) for all \( i \in \mathcal{I} \) or \( z_j \notin C_j^X(z, \alpha_j) \) for some \( j \notin \mathcal{I} \). Q.E.D.

Reny pointed out (Corollary 3.4) that if each \( u_i \) is lower semicontinuous in the strategies of the other players, then the game is necessarily payoff secure because for each \( x \), \( \epsilon > 0 \), and \( i \), player \( i \) can use \( x_i \) to secure \( u_i(x) - \epsilon \).

**Definition 8.4:** We say that \( u_i \) is multiply lower semicontinuous in the strategies of the other players if, for every \( x \in X \), there exists a finite closed cover \( F^1, \ldots, F^j \) of a neighborhood of \( x \) such that for each \( j \), there is some \( y_j^i \in X_i \) satisfying \( u_i(y_j^i, x_{-i}) \geq u_i(x) \) and the function \( z_{-i} \mapsto u_i(y_j^i, z_{-i}) \) restricted to \( F^j \) is lower semicontinuous.

The universal restriction operator is given by \( X_i(x) = X_i \) for all \( i \) and \( x \in X \). We drop the qualifier \( X \) for this particular restriction operator. Similarly, it makes sense to drop “multiply” if all the relevant closed covers have a single element. The next result follows from Proposition 8.3.
COROLLARY 8.5: If each $u_i$ is multiply lower semicontinuous in the strategies of other players, then it is multiply payoff secure. If in addition $G$ is quasiconcave and reciprocally upper semicontinuous, then there exists a Nash equilibrium.

Bagh and Jofre (2006) showed that better reply security is implied by payoff security and a condition that is weaker than reciprocal upper semicontinuity. We have not found a suitable analog of that concept. Carmona (2009) showed that existence of equilibrium is implied by a weakening of payoff security and a weak form of upper semicontinuity. We do not know of analogs of these conditions in our setting.

9. CONCLUSION

We have weakened the hypotheses of Reny’s theorem in several directions, achieved a briefer\(^3\) proof, and also extended his notions of payoff security and reciprocal upper semicontinuity. The generalized theorem has been used to prove the existence of pure Nash equilibrium in a new class of finite games, the diagnosable games, which includes examples in the spirit of locations models studied by Schelling (1971, 1972). Our result implies the Nishimura and Friedman (1981) existence theorem, which was one of the two earlier results not encompassed by Reny’s theorem, the other being Cournot equilibrium with fixed costs.

The concepts introduced in Section 3 were inspired in part by the hope of achieving a result that could be used to prove the existence result of Novshek (1985), which is the most refined result asserting the existence of Cournot equilibrium in the stream of literature following McManus (1962, 1964), Roberts and Sonnenschein (1976), and Szidarovzky and Yakowitz (1977). Specifically, it often happens that the set of quantities that allow a firm to obtain at least a certain level of profits will not be convex because it includes both increases and reductions to zero. Introducing a restriction operator allows the analyst to restrict attention to one possibility or the other at each point.

A proof of Novshek’s result by applying our result can be accomplished for the case of two firms, and was presented in an earlier version of this paper, but extending the argument to an arbitrary number of firms proved to be quite difficult. This seems to be related to the fact that although the general case is a game of strategic substitutes, in the sense of Bulow, Geanakoplos, and Klemperer (1985), the case of two firms can be construed as a game of strategic complements if the ordering of one of the two firm’s sets of quantities is reversed.

Subsequent literature related to Novshek’s result has gone in a different direction. Kukushkin (1994) presented a brief proof of the general case. As

\(^3\)Prokopovych (2011) gave a nice short proof for quasiconcave better reply secure games whose sets of pure strategies are metric spaces.
with various proofs of topological fixed point theorems, Kukushkin’s argument combines a nontrivial combinatoric result with a straightforward limiting argument. Kukushkin (2004) gave a more general treatment of finite games in which there is an ordering of strategy profiles that is increased by best response dynamics. Huang (2002), Dubey, Haimanko, and Zapechelnyuk (2006), and Kukushkin (2007) developed an approach to games of strategic substitutes and complements that generalizes the notion of a potential game. Jensen (2010) introduced a class of games that allows the methods of all these authors to be applied more broadly. In all of this literature a key feature is the presence of useful univariate aggregators.

There are many issues related to games on subsets that could be pursued. For example, we do not know of an algorithm for computing a Nash equilibrium of a location game other than exhaustive search. Since the size of the set of pure strategy profiles for the location model is an exponential function of the data required to define an instance, there is the question of whether there is a polynomial time algorithm for computing an equilibrium and, if not, what can be said about the computational complexity of this problem. The class of games on subsets is evidently quite rich and might have a number of applications that are not apparent at present. Even considering only the location model, an examination of our methods quickly reveals that the setup could be varied in many directions.

REFERENCES


School of Economics, University of Queensland, Level 6 Colin Clark Building, St. Lucia, QLD 4072 Australia; a.mclennan@economics.uq.edu.au,