Fixed Points of Parameterized Perturbations

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Abstract

Let $X$ be a convex subset of a locally convex topological vector space, let $U \subset X$ be open with $\overline{U}$ compact, let $F : \overline{U} \to X$ be an upper semicontinuous convex valued correspondence with no fixed points in $\overline{U} \setminus U$, let $P$ be a compact absolute neighborhood retract, and let $\rho : \overline{U} \to P$ be a continuous function. We show that if the fixed point index of $F$ is not zero, then there is a neighborhood $V$ of $F$ in the (suitably topologized) space of upper semicontinuous convex valued correspondences from $\overline{U}$ to $X$ such that for any continuous function $g : P \to V$ there is a $p \in P$ and a fixed point $x$ of $g(p)$ such that $\rho(x) = p$. This implies that no normal form game satisfies the conditions specified in Section 4.6 of Levy (2013).

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1 Introduction

The effect of perturbing a parameter—comparative statics—is, of course, a familiar and important issue in economic analysis. Perfection of a single Nash equilibrium (Selten (1975)) is defined by requiring that at least some perturbations in a given class give rise to perturbed games that have nearby equilibria. Roughly, Kohlberg and Mertens (1986) define strategic stability of a set of equilibria by requiring that for all sufficiently small

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perturbations, the perturbed games have equilibria near the set. This note presents a
topological result concerning the behavior of such nearby equilibria when there is a func-
tion from a neighborhood of the relevant set of equilibria to the space of perturbations.

The situation of interest arose in Levy (2013), which presented two examples of dis-
counted stochastic games with uncountable state spaces that do not have stationary
equilibria, one with deterministic transitions and one in which the transitions are ab-
solutely continuous with respect to a reference measure. Levy and McLennan (2014)
point out that the analysis of the second example is flawed, and they provide a different
example of a stochastic discounted game with absolutely continuous transitions, relative
to a reference measure, that does not have a stationary equilibrium. For this example
the underlying phenomenon generating nonexistence is drawn from analysis rather than topology.

Section 4.6 of Levy (2013) explains that the construction could be successfully exe-
cuted if certain perturbations of a “base” normal form game satisfied certain conditions.
The base game used in the flawed construction, which is due to Kohlberg and Mertens
(1986), has a set $S$ of Nash equilibria that is homeomorphic to a circle, and it is desired
that for each $\varepsilon > 0$ there is a map from the set of equilibria to the space of $\varepsilon$-perturbations
such that the perturbed game associated with each $\sigma \in S$ does not have any equilibria
near $\sigma$. It turns out that the perturbations given by Levy do not have this property.

A stochastic game of the desired sort could be constructed if another system of
perturbations of the Kohlberg and Mertens’ example satisfied the desired condition, or if
a different base game had perturbations with the properties laid out in Levy’s Section 4.6.
The result given here (already stated in the abstract) implies that in fact this situation
cannot be attained. We have yet to find other economic applications of this result, but it
is conceptually intriguing and aesthetically appealing (at least in the eyes of the author)
so perhaps it will eventually have other interesting consequences.

The remainder has the following organization. The next section reviews relevant
material from the theory of fixed points. Section 3 states the main result, and Section 4
explains its consequences for Levy (2013). The proof is given in Section 5.

## 2 Fixed Point Theory

This section gives a highly condensed account of relevant aspects of the topological theory
of fixed points. If $X$ and $Y$ are sets and $F : X \to Y$ is a set valued function, let $\text{Gr}(F) =
\{(x, y) \in X \times Y : y \in F(x)\}$ be the graph of $F$, and let $\mathcal{F}(F) = \{x \in X : x \in F(x)\}$
be its set of fixed points.
Let $X$ be a topological space. The space $X$ has the fixed point property if $\mathcal{F}(f) \neq \emptyset$ whenever $f : X \to X$ is a continuous function. It is contractible if there is a continuous function $c : X \times [0,1] \to X$ with $c(x,0) = x$ and $c(x,1) = c(x',1)$ for all $x, x' \in X$. Whether every nonempty compact contractible metric space has the fixed point property was an open question for several years, but Kinoshita (1953) (cf. Section 7.1 of McLennan (2012)) constructed an example of a compact contractible $X \subset \mathbb{R}^3$ and a continuous $f : X \to X$ without any fixed points. Thus the theory of fixed points requires some additional hypothesis on the spaces it considers.

If $A \subset X$ and $r : X \to A$ is continuous with $r(a) = a$ for all $a \in A$, then $r$ is said to be a retraction. If such an $r$ exists, then $A$ is a retract of $X$. A key intuition is that a neighborhood $U \subset A$ of a point $a$ may, in certain senses, inherit the topological simplicity of $r^{-1}(U)$. The space $X$ is an absolute neighborhood retract (ANR) (for metric spaces) if it is metrizable and, whenever $c : X \to Y$ is an embedding of $X$ in a metric space $Y$ and $c(X)$ is closed, $c(X)$ is a retract of some neighborhood of itself. The class of ANR’s includes closed convex subsets of locally convex vector spaces as well as simplicial complexes and smooth manifolds (McLennan (2012) Ch. 7). The Hawaiian earring $\bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^2 : \|(x - \frac{1}{n}, y)\| = \frac{1}{n}\}$ is an example of a space that is not an ANR. The fixed point theorem of Eilenberg and Montgomery (1946) implies that every nonempty compact contractible ANR has the fixed point property.

For a finite dimensional motivation of the fixed point index, consider a smooth manifold $M$. Let $U \subset M$ be open with $\overline{U}$ compact. If $f : \overline{U} \to M$ is a smooth function such that $\mathcal{F}(f) \subset U$ and $\text{Id}_{T_pM} - Df(p) : T_pM \to T_pM$ is nonsingular for all $p \in \mathcal{F}(f)$, then the index of $f$ is the number of $p \in \mathcal{F}(f)$ such that the determinant of $\text{Id}_{T_pM} - Df(p)$ is positive minus the number of $p \in \mathcal{F}(p)$ such that this determinant is negative.

The theory of the fixed point index extends this notion to a very high level of generality, by taking its main properties as axioms. An index admissible correspondence is an upper semicontinuous contractible valued correspondence $\mathcal{F} : \overline{U} \to X$ where $X$ is an ANR, $U \subset X$ is open, $\overline{U}$ is compact, and $\mathcal{F}(F) \subset U$. (That is, $\mathcal{F}(F) \cap (\overline{U} \setminus U) = \emptyset$.) The fixed point index is an assignment of an integer $\Lambda(F)$ to each such $F$ with the following properties:

(Normalization) If $X$ is nonempty and compact and $c : X \to X$ is a constant function, then $\Lambda(c) = 1$.

(Additivity) If $F : \overline{U} \to X$ is index admissible, $U_1, \ldots, U_r$ are disjoint open subsets of $U$, and $F$ has no fixed points in $\overline{U} \setminus (U_1 \cup \ldots \cup U_r)$, then $\Lambda(F) = \sum_{i=1}^{r} \Lambda(F|_{\overline{U}_i})$.

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1Here $T_pM$ is the tangent space of $M$ at $p$. 

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(Continuity) If \( F : \overline{U} \to X \) is index admissible, then \( \text{Gr}(F) \) has a neighborhood \( W \subset \overline{U} \times X \) such that \( \Lambda(F') = \Lambda(F) \) for all index admissible \( F' : \overline{U} \to X \) with \( \text{Gr}(F') \subset W \).

(Multiplication) If \( X \) and \( Y \) are ANR’s, \( U \subset X \) and \( V \subset Y \) are open, \( \overline{U} \) and \( \overline{V} \) are compact, \( F : \overline{U} \to X \) and \( G : \overline{V} \to Y \) are index admissible correspondences, and \( F \times G : \overline{U} \times \overline{V} \to X \times Y \) is the correspondence that takes \((x, y)\) to \( F(x) \times G(y) \), then \( \Lambda(F \times G) = \Lambda(F) \cdot \Lambda(G) \).

(Commutativity) If \( X \) and \( X' \) are ANR’s, \( V \subset U \subset X \) and \( V' \subset U' \subset X' \) with \( U \), \( V \), \( U' \), and \( V' \) open, \( \overline{U} \) and \( \overline{U'} \) are compact, \( f : \overline{U} \to X \) and \( f' : \overline{U'} \to X \) are continuous functions with \( f(\overline{V}) \subset \overline{U} \) and \( f'(\overline{V'}) \subset \overline{U'} \), \( f \circ f'|_{\overline{V'}} \) and \( f' \circ f|_{\overline{V}} \) are index admissible, and \( f \) maps the fixed points of \( f' \circ f|_{\overline{V}} \) onto the fixed points of \( f \circ f'|_{\overline{V'}} \), then \( \Lambda(f' \circ f|_{\overline{V}}) = \Lambda(f \circ f'|_{\overline{V'}}) \).

The fixed point index resulted from the evolution of the theory of fixed points during the first half of the twentieth century; it is commonly attributed to Leray and Schauder. The axiomatic characterization is due to O’Neill (1953). McLennan (2008) is a brief but somewhat more informative introduction. Brown (1971) and McLennan (2012) are comprehensive treatments of existence and uniqueness of the index; the first of these is from the point of view of pure mathematics, while the second emphasizes issues (such as correspondences) of importance to economics. Dugundji and Granas (2003) is another comprehensive reference.

We now make several remarks concerning the fixed point index.

Suppose that \( F : \overline{U} \to X \) is index admissible. If \( F \) has no fixed points, then Additivity gives \( \Lambda(F|_{\emptyset}) = \Lambda(F) = \Lambda(F|_{\emptyset}) + \Lambda(F|_{\emptyset}) \) and thus \( \Lambda(F) = 0 \). Conversely, if \( \Lambda(F) \neq 0 \), then \( F \) is not essential.

Suppose that \( K \subset \mathcal{F}(F) \) is compact and has a neighborhood that contains no other fixed points of \( F \). The index of \( K \) (for \( F \)) is \( \Lambda(F|_{\overline{V}}) \) where \( V \) is an open set with \( V \cap \mathcal{F}(F) = \overline{V} \cap \mathcal{F}(F) = K \); Additivity implies that this number does not depend on the choice of \( V \). When \( \mathcal{F}(F) \) has finitely many connected components, viewing the index as an assignment of an integer to each component is intuitively appealing.

The set \( K \) is essential (for \( F \)) (Fort (1950), Kinoshita (1952), O’Neill (1953)) if for any neighborhood \( Z \) of \( K \) there is a neighborhood \( W \subset \overline{U} \times X \) of \( \text{Gr}(F) \) such that each index admissible \( F' : \overline{U} \to X \) with \( \text{Gr}(F') \subset W \) has a fixed point in \( Z \). Otherwise \( K \) is inessential. Any neighborhood of \( \text{Gr}(F') \) contains the graph of a continuous function (Kakutani (1941), Mas-Colell (1974), McLennan (1989), cf. Ch. 9 of McLennan (2012)); we will refer to this result as the approximation theorem. It implies that \( K \) is essential or not according to whether all continuous functions near \( F \) have fixed points near \( K \).
If $K$ is inessential, then its index is zero\(^2\) by Continuity, because the index of nearby perturbations is zero.

Suppose that $X$ is a compact ANR, $U \subset X$ is open, and $\overline{U}$ is compact. Let $\mathcal{U}$ be the set of upper semicontinuous contractible valued correspondences $F : \overline{U} \to X$. We endow $\mathcal{U}$ with the topology that has a base consisting of the sets of the form $\{ F \in \mathcal{U} : \text{Gr}(F) \subset W \}$ where $W \subset \overline{U} \times X$ is open; Continuity asserts precisely that the index is continuous with respect to this topology. If $Y$ is a regular\(^3\) topological space and $H : \overline{U} \times Y \to X$ is a correspondence whose values are compact and contractible, then $H$ is upper semicontinuous if and only if $y \mapsto H(\cdot, y)$ is a continuous function from $Y$ to $\mathcal{U}$ (Theorem 5.4.3 of McLennan (2012)). We refer to this as the homotopy principle because the most important case is $Y = [0, 1]$. In this way we see that the index is invariant under homotopy: more generally, if each $H(\cdot, y)$ is index admissible, then Continuity implies that $\Lambda(H(\cdot, y))$ is a locally constant function of $y$, hence constant on each connected component of $Y$.

Let $C \subset \mathbb{R}^m$ be nonempty, compact, and convex. If $F : C \to C$ is upper semicontinuous and convex valued, then Continuity and the approximation theorem imply that $\Lambda(F) = \Lambda(f)$ for some continuous function $f : C \to C$. Since $f$ is homotopic to a constant function, the homotopy principle, Continuity, and Normalization imply that $\Lambda(f) = 1$. Since $\Lambda(F) \neq 0$ implies $\mathcal{F}(F) \neq \emptyset$, we see that the theory of the fixed point index subsumes the Kakutani fixed point theorem. If $\mathcal{F}(F)$ is the union of finitely many connected components, then each of these has a neighborhood containing no other fixed points, so its index is well defined, and of course Additivity implies that the sum of these indices is one.

### 3 The Result

Let $X$ be a convex subset of a locally convex topological vector space. Automatically $X$ is an ANR (e.g., Proposition 7.4.1 of McLennan (2012)). Let $U \subset X$ be open with $\overline{U}$ compact, and let $\mathcal{U}$ be the set of upper semicontinuous convex valued correspondences $F : \overline{U} \to X$, endowed with the topology described above. (More precisely, it has the relative topology it inherits as a subspace.) Let $P$ be a compact ANR, thought of as a space of parameters. The main result is:

**Theorem 1.** If $F \in \mathcal{U}$ is index admissible, $\Lambda(F) \neq 0$, and $\rho : \overline{U} \to P$ is a continuous

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\(^2\)The converse—that $K$ is inessential if its index is zero—holds for connected $K$ when $X$ is convex and finite dimensional and $F$ is convex valued (Theorem 14.5.3 of McLennan (2012)).

\(^3\)A topological space is regular if every neighborhood of a point contains a closed neighborhood.
function, then there is a neighborhood \( V \subset U \) of \( F \) such that for any continuous \( g : P \to V \) there is a \( p \in P \) such that \( \rho^{-1}(p) \cap \mathcal{F}(g(p)) \neq \emptyset \).

Remark: Since a perturbation \( g : P \to V \) parameterized by \( P \) induces a perturbation \( g \circ \rho \) parameterized by \( \overline{U} \), it might seem that the result could be simplified by identifying \( P \) with \( \overline{U} \), but this is not without loss of generality because \( \overline{U} \) need not be an ANR. On the other hand, the proof does depend on the fact that \( P \times X \) is an ANR (Propositions 7.4.1 and 7.4.3 of McLennan (2012)). As with the the fixed point property for compact contractible sets, it is not easy to give an example showing that the hypothesis that \( P \) is an ANR is indispensable, but the proof applies fixed point theory to \( P \times X \), so this seems quite likely.

4 Application

Section 4.6 of Levy (2013) specifies conditions on a “base” game that are sufficient for the method of that paper to yield an example of a stochastic game with absolutely continuous transitions that has no stationary equilibrium. In this section we explain why no game satisfies those conditions. Let \( G \) be a finite strategic form game, and let \( \text{NE} \) be its set of Nash equilibria.

**Corollary 1.** If \( P \subset \text{NE} \) is an absolute neighborhood retract, \( U \) is a neighborhood of \( \text{NE} \) in the space of mixed strategy profiles, and \( \rho : U \to P \) is a retraction, then there is a neighborhood \( W \) of \( G \) in the space of games (for the given strategic form) such that for any continuous \( h : P \to W \) there is some \( e \in P \) such that \( \rho^{-1}(e) \) contains a Nash equilibrium of \( h(e) \).

**Proof.** To obtain this from Theorem 1, let \( X \) be the set of mixed strategy profiles of \( G \), let \( F \) be its best reply correspondence, and for \( e \in P \), let \( g(e) \) be the best response correspondence of \( h(e) \). (The map taking each game to its best response correspondence is easily shown to be continuous when the space of correspondences has the topology described in the last section.)

For a positive integer \( n \) let \( C^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \} \) be the boundary of the hypercube \([-1, 1]^{n+1}\). By a **facet** of \( C^n \) we will mean a subset of the forms \( \{ x \in C^n : x_i = 1 \} \) and \( \{ x \in C^n : x_i = -1 \} \). The following conditions are slightly weaker than those specified by Levy (2013):

1. \( \text{NE} \) contains a unique hyperstable set \( H \).
2. \( H \) is connected but not nullhomotopic.
(3) For some positive integer $n$ there exist:

(a) a continuous embedding $\psi : C^n \to H$ that is not nullhomotopic in $H$;

(b) a retraction $\tilde{\rho} : \text{NE} \to \psi(C^n)$;

(c) for all $\varepsilon > 0$, a continuous function $\Gamma_\varepsilon$ from $C^n$ to the $\varepsilon$-neighborhood of $G$ (in the space of games with the same pure strategy sets) such that for each facet $E$ of $[-1,1]^{n+1}$, any equilibrium of any game in $\Gamma_\varepsilon(E)$ is in the $\varepsilon$-neighborhood of $\tilde{\rho}^{-1}(\psi(\varepsilon))$.

The precise definition of hyperstability (Kohlberg and Mertens (1986)) will not be required. It involves a robustness condition on compact subsets of NE that we will refer to as quasihyperstability. (Kohlberg and Mertens do not use this term.) A compact subset of NE is necessarily quasihyperstable if a subset is quasihyperstable. A compact subset of NE is hyperstable if it is a minimal (with respect to set inclusion) quasihyperstable set. Kohlberg and Mertens show that any quasihyperstable set contains a hyperstable set. Govindan and Wilson (2005) show that a connected component of NE is essential if and only if it satisfies a condition called uniform hyperstability that is slightly stronger than quasihyperstability.

We note that (1), (2), and (a) and (b) of (3) are satisfied by the base game used by Levy (2013) (which is taken from Kohlberg and Mertens (1986)) because its set of Nash equilibria is homeomorphic to $C^1$, and was shown by Kohlberg and Mertens to be the unique hyperstable set. Assuming that (1), (2), and (3) (including (c)) hold, we will show that Corollary 1 is violated.

As a semi-algebraic set, NE is triangulable (e.g., Bochnak et al. (1987)) and thus the union of finitely many connected components. One of these contains $H$. Each other connected component does not contain a hyperstable set, hence is not quasihyperstable, hence is not uniformly hyperstable, hence is inessential, and consequently its index is zero. As we saw above, the index of NE is one, and Additivity implies that the index of the component containing $H$ is also one.

Since NE is triangulable, and a simplicial complex is an ANR (e.g., Propositions 7.3.2 and 7.4.1 of McLennan (2012)) there is a retraction of a neighborhood $U$ of NE onto NE. The composition of this with $\tilde{\rho}$ gives an extension $\rho$ of $\tilde{\rho}$ to $U$. Since $\psi$ is an embedding, for each facet $E$ of $C^n$, $\tilde{\rho}^{-1}(\psi(E))$ and $\tilde{\rho}^{-1}(\psi(-E))$ are disjoint compact sets, so for sufficiently small $\delta$ the distance between these two sets is at least $2\delta$. By replacing $U$ with a smaller neighborhood of NE we can insure that for each $E$, $\rho^{-1}(\psi(E))$ is contained in the $\delta$-ball surrounding $\tilde{\rho}^{-1}(\psi(E))$. 

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Now suppose that for each \( \varepsilon > 0 \) there is a continuous function \( \Gamma_\varepsilon \) from \( C^n \) to the \( \varepsilon \)-neighborhood of \( G \), as per (c) of (3). The upper semicontinuity of the Nash equilibrium correspondence implies that for sufficiently small \( \varepsilon > 0 \), \( U \) contains the set of Nash equilibria of \( \Gamma_\varepsilon(c) \) for any \( c \in C^n \). If \( \varepsilon < \delta \), \( c \) is an element of the facet \( E \), and (as per (c)) the set of Nash equilibria of \( \Gamma_\varepsilon(c) \) is contained in the \( \varepsilon \)-neighborhood of \( \tilde{\rho}^{-1}(\psi(-E)) \), then there are no equilibria in \( \rho^{-1}(\psi(c)) \). But Corollary 1 implies that this cannot be the case for all \( c \in C^n \).

Evidently this contradiction could be obtained from hypotheses that are weaker than (1)-(3) in several ways. In particular, the assumption that \( H \) and \( \psi \) are not nullhomotopic plays no role, and the retraction \( \tilde{\rho} \) need only be defined on the connected component of NE containing \( H \).

5 Proof

In preparation for the proof, we have the following topological fact.

**Lemma 1.** If \( X \) is a compact regular space, \( Y \) is a convex subset of a locally convex topological vector space, and \( F : X \to Y \) is an upper semicontinuous compact convex valued correspondence, then any neighborhood \( W \) of \( \text{Gr}(F) \) contains an open neighborhood \( W' \) such that \( \{ y : (x, y) \in W' \} \) is convex for each \( x \in X \).

**Proof.** For each \( x \in X \) and \( y \in F(x) \) the definition of the product topology, continuity of addition in the TVS containing \( Y \), and regularity, give a closed neighborhood \( D_y \) of \( x \) and an open neighborhood \( V_y \) of the origin such that \( D_y \times (Y \cap (y + V_y + V_y)) \subset W \). Since \( F(x) \) is compact, it is contained in \( \bigcup_{i=1}^k y_i + V_{y_i} \) for some \( y_1, \ldots, y_k \). Setting \( C_x = \bigcap_i D_{y_i} \) and \( U_x = (F(x) + \bigcap_i V_{y_i}) \cap Y \) gives a closed neighborhood of \( x \) and a convex open neighborhood of \( F(x) \) such that \( C_x \times U_x \subset W \). Regularity and the definition of upper semicontinuity imply that we may achieve \( \text{Gr}(F) \cap (C_x \times X) \subset C_x \times U_x \) by replacing \( C_x \) with a smaller closed neighborhood of \( x \). Now choose \( x_1, \ldots, x_\ell \) such that \( \bigcup_j C_{x_j} = X \), and let

\[
W' = \bigcup_{x \in X} \left( \{ x \} \times \bigcap_{x \in C_{x_j}} U_{x_j} \right).
\]

Evidently \( W' \) contains \( \text{Gr}(F) \) by construction and is open because each \( x \in X \) has a neighborhood whose points \( x' \) all satisfy \( \{ j : x' \in C_{x_j} \} \subset \{ j : x \in C_{x_j} \} \). \( \square \)

**Proof of Theorem 1.** For an open set \( W \subset \overline{U} \times X \) containing \( \text{Gr}(F) \), let \( V \) be the set of \( F' \in U \) such that \( \text{Gr}(F') \subset W \). We can replace \( W \) with \( W \setminus \{ (x, x) : x \in \overline{U} \setminus U \} \), so that each element of \( V \) has no fixed points outside of \( U \) and is consequently index admissible.
Continuity allows us to choose $W$ so that $\Lambda(F') = \Lambda(F) \neq 0$ for all $F' \in V$. By the result above we may assume that $\{ y : (x, y) \in W \}$ is convex for each $x$. The approximation theorem implies that there is a continuous function $f : \overline{U} \rightarrow X$ with $\text{Gr}(f) \subset W$.

Let $g : P \rightarrow V$ be continuous. The homotopy principle implies that we may regard $g$ as an upper semicontinuous convex valued correspondence from $P \times \overline{U}$ to $X$ whose graph is contained in $P \times W$. Our objective is to show that there is a $(p, x)$ with $x \in g(p, x)$ and $p(x) = p$. By the approximation theorem, any neighborhood of the graph of $g$ contains the graph of a continuous function, and if there was no such $(p, x)$, then this would also be the case if $g$ was replaced by a sufficiently nearby function. Therefore we may assume that $g : P \times \overline{U} \rightarrow X$ is a continuous function.

Setting $U' = P \times U$, let $\varphi : \overline{U} \rightarrow P \times X$ and $\varphi' : \overline{U} \rightarrow X$ be the functions

$$\varphi(x) = (p(x), f(x)) \quad \text{and} \quad \varphi'(p, x) = x.$$ 

Let $Z = U \cap f^{-1}(U)$ and $Z' = P \times Z$. We have $f(Z) \subset \overline{U}$, $\varphi(Z) \subset \overline{U}$, and $\varphi'(Z') \subset \overline{U}$, so $\varphi' \circ \varphi|_{\overline{U}} = f|_{\overline{U}}$, and $\varphi \circ \varphi'|_{\overline{U}}$ is the map $(p, x) \mapsto (p(x), f(x))$. Therefore $\mathcal{F}(\varphi' \circ \varphi|_{\overline{U}}) = \mathcal{F}(f)$ because $\mathcal{F}(f) \subset Z$, and $\mathcal{F}(\varphi \circ \varphi'|_{\overline{U}}) = \{(p(x), x) : x \in \mathcal{F}(f)\}$. In particular, $\varphi' \circ \varphi|_{\overline{U}}$ and $\varphi \circ \varphi'|_{\overline{U}}$ are index admissible, and $\varphi$ maps $\mathcal{F}(\varphi' \circ \varphi|_{\overline{U}})$ onto $\mathcal{F}(\varphi \circ \varphi'|_{\overline{U}})$, so Commutativity and Additivity give $\Lambda(\varphi' \circ \varphi|_{\overline{U}}) = \Lambda(\varphi' \circ \varphi|_{\overline{U}}) = \Lambda(f|_{\overline{U}}) = \Lambda(f) \neq 0$.

Let $h : P \times \overline{U} \times [0, 1] \rightarrow X$ be the homotopy $h(p, x, t) = (1 - t)f(x) + tg(p, x)$. Since $f(x), g(p, x) \in \{y : (x, y) \in W\}$ for all $p$ and $x$, the graph of each $h_t$ is contained in $P \times W$. (We follow the standard notational convention, writing $h_t$ in place of $h(\cdot, \cdot, t)$.) Let $\eta : \overline{Z} \times [0, 1] \rightarrow P \times X$ be the homotopy $\eta(p, x, t) = (p(x), h(p, x, t))$; note that $\eta_t = \varphi \circ \varphi'|_{\overline{U}}$ and $\eta_t(p, x) = (p(x), g(p, x))$. For each $t$, if $(p, x)$ is a fixed point of $\eta_t$, then $x$ is a fixed point of $h_t(p, \cdot)$ and consequently an element of $U$. Therefore the fixed points of each $\eta_t$ are contained in $P \times U$, so $\eta_t$ is index admissible. Now the homotopy principle and Continuity imply that $\Lambda(\eta_t) = \Lambda(\eta_0) \neq 0$, and consequently $\eta_1$ has a nonempty set of fixed points. That is, for some $p$ and $x$, $x \in \rho^{-1}(p) \cap \mathcal{F}(g(p))$. 

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