TRUTHFUL IMPLEMENTATION AND AGGREGATION IN RESTRICTED DOMAINS

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Abstract. We consider a social choice setting where agents have quasi-linear utilities over social alternatives and a transferable commodity, and study three properties that a social choice function may possess: truthful implementation (in dominant strategies); monotonicity in differences; and affine maximization, which has weak and strong forms. We introduce the notion of a flexible domain of preferences and study which of these conditions implies which others when the domain is flexible. We give examples showing that the result stated by Roberts (1979) is flawed; our main result is a strict generalization of the corrected version of that result. Flexibility holds, and the theorem is not vacuous, if the domain of valuation profiles is restricted to the space of continuous functions defined on a topological space, or the space of piecewise linear functions defined on an affine space, or the space of smooth functions defined on a compact differentiable manifold. Some illustrative economic applications are provided.

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1. Introduction

The relationship between the implementability of a social choice function and other properties that this aggregation device may possess...
is a central theme in the mechanism design and social choice literatures. In reaction to the fundamental result of Gibbard (1973) and Satterthwaite (1975), a natural direction for investigation is to restrict the domain of profiles, aiming either at similar results with weaker hypotheses or positive results\(^1\). The most obvious restrictions on the domain of the social choice function are obtained by restricting each agent to have a preference drawn from some subset of the space of strict preference orderings\(^2\), and our work considers only domain restrictions of this sort.

A particularly important possibility is that side payments may be feasible. Such environments occur naturally in connection with, for example, public good provision models with projects of variable size (Green and Laffont (1977), Green and Laffont (1979), Laffont and Maskin (1980)), cost sharing agreements (Moulin (1988) and more recently Moulin (2008)), allocation models of pollution permits and other common divisible resources (Dasgupta, Hammond, and Maskin (1980), Duggan and Roberts (2002), Montero (2008)), and so forth.

Roberts (1979) considers an environment in which an outcome of the social choice process consists of an element of a finite set \(A\) of social alternatives and a vector of side payments. Roberts restricts agents to have preferences that are quasilinear: an agent’s utility is the sum of a utility given by the societal alternative and the monetary transfer she receives. In the literature related to Roberts’ work a social choice function (SCF) is a function from the domain of preference profiles to \(A\). Such an SCF is said to be truthfully implementable (in dominant strategies) if it can be combined with a payment function, mapping preference profiles to vectors of side payments, to create a social choice function, in the more complete sense of Gibbard and Satterthwaite, for which truth telling is weakly dominant.

\[^1\]The dictatorial conclusion of the Gibbard-Satterthwaite Theorem holds even if the domain of preferences is restricted to the space of all continuous functions defined on a metric space; cf. Barberà and Peleg (1990). In rich domains consisting of strict preferences, Dasgupta, Hammond, and Maskin (1979) show that a social choice function is truthfully implementable in dominant strategies if and only if it is monotonic, in the sense that, for any two utility profiles, if alternative \(a\) is selected by the choice function under the first utility profile but \(a\) is not selected under the second profile, then there exist an alternative \(b\) and an agent in the society that weakly prefers \(a\) to \(b\) at the first situation but strictly prefers \(b\) to \(a\) at the second one. See also Maskin (1985). This monotonicity condition is sometimes referred to as strong positive association; cf. Muller and Satterthwaite (1977).

\[^2\]Single peakedness (Black (1958)) is an example of a well studied domain restriction that does not have this form.
An SCF is an affine maximizer if it maximizes a social welfare function that is a weighted sum of the agents’ utilities plus a function that may be thought of as representing societal values, such as externalities or the welfare of future generations, that are not captured by individual preferences. In this definition the weights on the individual utilities are required to be nonnegative, with at least one individual having a positive weight. If the weights are all positive then we say that the SCF is a strong affine maximizer.

Roberts (1979) asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted—that is, the agents can attach any values to the elements of $A$—then an SCF is truthfully implementable in dominant strategies if and only if it is an affine maximizer, and that in turn these conditions hold if and only if a third condition called positive association of differences (PAD) holds. However, one of our contributions is to show that this is not quite correct: a strong affine maximizer can be truthfully implemented, and a truthfully implementable SCF is an affine maximizer, but Example 2.4 gives an affine maximizer that is not truthfully implementable. We also give a corrected version of the result that exactly characterizes truthful implementability.

The complication arises when some agents’ preferences have zero weight in the affine combination. At each profile of types for the agents with positive weight there is an induced SCF whose domain is the set of type profiles for the agents with zero weight and whose range is the set of alternatives maximizing the affine functional. If the given SCF is truthfully implementable, then so is this derived SCF, so it must be an affine maximizer when its image has three or more elements. Developing this condition recursively leads to the notion of a lexicographic affine maximizer, and the corrected result asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted, then truthfully implementability, PAD, and lexicographic affine maximization are equivalent.

The assumption that $A$ is finite is unnatural in many applications, e.g., typically the quantity of a public good can be any real number in some range. Previous characterization results of implementable choice functions via affine maximizers are not appropriate to handle such situations. When $A$ is infinite the equivalence asserted by Roberts’ theorem continues to hold in a formal sense, but one can easily construct profiles for which affine maximization is undefined, so the result in that case is vacuous. In order to have a meaningful generalization of Roberts’ result in this direction one must introduce restrictions on the domain. One of our major objectives is to extend Roberts’ theorem

\[3\] In addition to Roberts (1979), see Lavi, Mu’alem, and Nisan (2009), Dobzinski and Nisan (2009), Mishra and Sen (2010) and Vohra (2009).
to the case in which the set of alternatives is a compact metric space and the admissible preferences are continuous functions on $A$.

Our strategy is to impose conditions on the spaces of individual preferences that insure that they are “rich enough” to support arguments leading to the conclusions of Roberts’ theorem (as corrected above) but are at the same time satisfied in a wide variety of applications. In Section 2 we introduce two conditions, *elevation of pairs* and *flexibility*. These conditions are rather technical, but at this point we can say that they have the following character: given two (or in some cases three) alternatives and two (or in some cases three) preferences, there is another preference that emphasizes the given alternatives because other alternatives become less desirable.

Instead of using PAD, we focus on a condition we call *monotonicity in differences*. For any two admissible profiles of preferences, monotonicity in differences requires that if alternative $a$ is selected by the choice function under the first profile and alternative $b$ is chosen under the second profile, then there exists at least one agent for whom the valuation difference between $b$ and $a$ is weakly greater in the second situation than in the first. Like PAD, monotonicity in differences is a collective condition insofar as it considers simultaneous changes of valuations for multiple agents. Unlike PAD, monotonicity in differences is not implied by the dominant strategies incentive constraints in every preference domain, nor does it necessarily imply truthful implementation or the affine maximization property in all domains.

Our main result, Theorem 1, states that if the domain of preferences allows elevation of pair and is flexible, and the image of the SCF has at least three elements, then it is truthfully implementable if and only if it is a lexicographic affine maximizer. In more detail, we establish the following implications. For all domains, lexicographic affine maximization implies truthful implementation and affine maximization implies monotonicity in differences. If the domain of profiles is a cartesian product of spaces of allowed preferences for the various agents, and each agent’s space allows elevation of pairs, then truthful implementability implies monotonicity in differences. If the domain of profiles is flexible, then monotonicity in differences implies affine maximization. These two results are stated in Section 2 and proved in Section 5; the influence of the arguments of Lavi, Mu‘alem, and Nisan (2009) will be evident. As we will see at the end of Section 2, these results combine to imply Theorem 1.

In Section 3 we study some of the implications of our main result. To this end we introduce another condition called *comprehensiveness*, and we show that comprehensiveness implies that the domain of preferences
allows elevations of pairs and is flexible. Recall that a lattice of real valued functions is a vector space of functions that is closed under the pointwise minimum and pointwise maximum operators. A resolving lattice for $A$ is a lattice with the additional property that for each $x \in A$ there is a function in the lattice whose value at $x$ is different from its value at every other point in $A$. The second key result of Section 3 shows that if a domain of valuations is a resolving lattice, then it is comprehensive. It is easy to see that the space of continuous functions on a compact metric space is a resolving lattice, as is the set of continuous piecewise affine functions on a convex subset of a Euclidean space. The space of $C^r$ functions on a $C^r$ manifold is not a lattice, so, at least from the point of view of the proper development of our techniques, one should inquire as to whether Roberts’ theorem also holds in this case. It turns out that it does. Because of some additional technical complications, the details are explained in the Appendix.

Section 4 presents several examples. One illustrates how comprehensiveness can hold in domains that are strict subsets of the unrestricted preference domain considered by Roberts (1979) even when the set of alternatives is finite, so our result is a strict generalization in this case. Incidentally, Mishra and Sen (2010) characterize neutral social choice functions via weighted welfare maximizers in open interval domains, still assuming that the alternative set $A$ is finite. The neutrality of a choice function may be a reasonable assumption for certain environments, but is violated whenever the choice function discriminates a priori among alternatives, as occurs for instance when the social objective function includes a function embodying societal values. On the other hand, as Mishra and Sen (2010) point out, neutrality is an essential component for restricting the domain to open bounded intervals.

We also present a simple example to illustrate the fact that, in certain domains, truthful implementation neither implies, nor is implied by, monotonicity in differences. This example also shows that a monotonic SCF is not necessarily an affine maximizer in every domain. A second example shows that truthful implementation also does not necessarily imply affine maximization when the image of the SCF has three or more alternatives. (This has also been shown by Mishra and Sen (2010).) Thus these conditions are not equivalent in every domain.

Section 5 presents two of the longer proofs, and Section 6 presents some final remarks. An Appendix shows how to extend the implications of our results to smooth valuations on a smooth compact manifold.

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4Our example draws inspiration from Bikhchandani, Chatterji, Lavi, Mu’alem, Nisan, and Sen (2006a).
2. The social choice setting and main result

There is a set $A$ of social alternatives, which can be finite or have any infinite cardinality. There is a finite set $N = \{1, \ldots, n\}$ of agents. For each $i \in N$ there is a space $T_i$ of types. The space of type profiles is $T = T_1 \times \cdots \times T_n$. A social choice function (SCF) is a function $f: T \to A$.

We fix such an $f$. The social decision process will pass from a vector $t \in T$ to $f(t)$ and a vector of monetary transfers to the agents.

It is customary in this literature to assume that $f$ is surjective, in conjunction with assumptions concerning the cardinality of $A$. Surjectivity of a SCF is also known as non imposition (e.g. Roberts (1979)) and citizen sovereignty (e.g. Dasgupta, Hammond, and Maskin (1979)). Here we do not impose this, instead imposing assumptions on the cardinality $|f(T)|$ of the image of $f$. While surjectivity may be philosophical significant in some settings, in the formal analysis it is a matter of convention rather than substance. Our framework is convenient because there will be derived SCF’s with smaller images.

Let $\mathcal{D}(A)$ be the space of functions $d: A \times A \to \mathbb{R}$ satisfying

$$d(a, b) + d(b, c) + d(c, a) = 0$$

for all $a, b, c \in A$. Elements of $\mathcal{D}(A)$ are called differences. Note that setting $a = b = c$ gives $d(a, a) = 0$, and in view of this, setting $b = c$ gives $d(b, a) = -d(a, b)$. We assume that for each $t_i \in T_i$ there is an associated difference $d_{t_i} \in \mathcal{D}(A)$. The interpretation is that $t_i \in T_i$ signifies (perhaps among other things) that for all $a, b \in A$ and $\tau \in \mathbb{R}$, agent $i$ is indifferent between having $a$ implemented and receiving a transfer of $\tau$ and having $b$ implemented with a transfer of $\tau + d_{t_i}(a, b)$. Thus, we are assuming that agents have quasi-linear preferences.

In most of the social choice literature an agent is characterized by a valuation, which is a function $v_i: A \to \mathbb{R}$. One may derive a difference from a valuation $v_i$ by setting $d_{v_i}(a, b) = v_i(a) - v_i(b)$. In fact one motivation for having general type spaces here is that the notation for the derivation of a difference from a type is the same as that for the derivation of a difference from a valuation, and much of the discussion revolves around the latter derivation, so the additional generality does not increase the notational burden. The significance of working with differences rather than valuations is, perhaps, more formal than real, but for reasons that are familiar from other mathematical endeavors, it is psychologically valuable. Specifically, it disciplines the analysis by systematically excluding certain extraneous information, which tends to result in greater simplicity and clarity.
In principle it is more correct to have general type spaces for the agent: it is better to derive restrictions on the information that agents report in the operation of the mechanism instead of imposing them a priori. As Roberts theorem was originally stated, the additional generality of abstract type spaces would have complicated the presentation, but the final result would have been the same. In our corrected version of the result the additional generality of general type spaces is more significant: given an affine maximizer, the derived mechanisms of the result the additional generality of general type spaces is implementable SCF’s with $|f(T)| \geq 3$, but we do not restrict the discussion to such SCF’s. Note that any constant SCF ($|f(T)| = 1$) is truthfully implementable. We say that $f$ is a binary implementable SCF if it is truthfully implementable in dominant strategies and $|f(T)| = 2$. Such SCF’s are characterized by cutoff differences.

**Proposition 1.** If $f(T) = \{x, y\} \subset A$, where $x \neq y$, then $f$ is truthfully implementable if and only if for all $i \in N$ and all $t_{-i} \in T_{-i}$, there exists $\delta^*_i(t_{-i}) \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that the following conditions hold:

(a) $t_i \in T_i$ and $d_{i,i}(x,y) > \delta^*_i(t_{-i})$ imply $f(t_i, t_{-i}) = x$.
(b) $t_i \in T_i$ and $d_{i,i}(x,y) < \delta^*_i(t_{-i})$ imply $f(t_i, t_{-i}) = y$.

**Proof.** If functions $\delta^*_1, \ldots, \delta^*_n$ satisfying the conditions above exist, then $f$ is truthfully implementable by virtue of the payment scheme in which $p_i(t) = 0$ except when $-\infty < \delta^*_i(t_{-i}) < \infty$, $d_{i,i}(x,y) \geq \delta^*_i(t_{-i})$, and $f(t) = x$, in which case $p_i(t) = \delta^*_i(t_{-i})$. On the other hand, if there do not exist $\delta^*_1, \ldots, \delta^*_n$ satisfying the conditions, then for some $i \in N$ and $t_{-i} \in T_{-i}$ there are $t_i, t'_i \in T_i$ such that $d_{i,i}(x,y) < d_{i,i}'(x,y)$, $f(t_i, t_{-i}) = x$, and $f(t'_i, t_{-i}) = y$. For any payment scheme, either $p_i(t_i, t_{-i}) \geq p_i(t'_i, t_{-i})$, in which case type $t_i$ will prefer to report that
her type is $t'_i$, or the opposite inequality holds, in which case type $t'_i$ will prefer to report that her type is $t_i$. □

For $t \in T$ let $d_t = (d_{t_1}, \ldots, d_{t_n})$. If $G : \mathbb{R}^n \to \mathbb{R}$ is weakly increasing, in the sense that $G(\delta') \geq G(\delta)$ whenever $\delta'_i \geq \delta_i$ for all $i$, we say that $G$ is strictly increasing at zero if for any $\delta \in \mathbb{R}^n$ with $G(\delta) = 0$, one has $G(\delta_i - \epsilon, \delta_{-i}) < 0 < G(\delta_i + \epsilon, \delta_{-i})$ for each $i$ and every $\epsilon > 0$. Suppose that this is the case, that $f(T) = \{x, y\}$, and that $f(t) = x$ for all $t$ such that $G(d_t(x, y)) > 0$ and $f(t) = y$ for all $t$ such that $G(d_t(x, y)) < 0$. For each $i$ and $t_{-i} \in T_{-i}$ we let $\delta^*_i(t_{-i})$ be the infimum of the set of $\delta_i$ such that $G(\delta_i, d_{t_{-i}}(x, y)) > 0$ if this set is nonempty, and otherwise we set $\delta^*_i(t_{-i}) = \infty$. The result above implies that $f$ is binary implementable.

Thus the class of binary implementable SCF’s is (in comparison with the affine maximizers we will see below) quite rich. And there are also binary implementable SCF’s that are not derived from this construction: in the following example $\delta^*_2(t_1)$ does not depend solely on $d_{t_1}$.

Example 2.2. Let $N = \{1, 2\}$ with $T_1 = \mathbb{R} \cup \{t'_1\}$, $T_2 = \mathbb{R}$, $d_{t_1}(x, y) = 0$, $d_{t'_1}(x, y) = t_1$ for all $t_1 \neq t'_1$, and $d_{t_2}(x, y) = t_2$ for all $t_2 \in T_2$. Let $\hat{f} : T \to A$ be a SCF defined as follows.

$$
\hat{f}(t'_1, t_2) = \begin{cases} 
  x : & \text{if } d_{t_2}(x, y) > 1; \\
  y : & \text{if } d_{t_2}(x, y) \leq 1;
\end{cases}
$$

$$
\hat{f}(t_1, t_2) = \begin{cases} 
  x : & \text{if } t_1 \neq t'_1, d_{t_1}(x, y) \geq 0, \text{ and } d_{t_2}(x, y) \geq 0; \\
  y : & \text{otherwise.}
\end{cases}
$$

The hypotheses of Proposition 1 are satisfied: if $t_2 \geq 0$ then $\delta^*_1(t_2) = 0$, and otherwise $\delta^*_1(t_2) = \infty$, while $\delta^*_2(t'_1) = 1$ and $\delta^*_2(t_1) = 0$ for all $t_1 \neq t'_1$.

Let $\Delta^{n-1} = \{ \sigma \in \mathbb{R}^n_+ \mid \sigma_1 + \cdots + \sigma_n = 1 \}$ be the $(n-1)$-dimensional simplex.

Definition 2.3. We say that $\hat{f} : T \to A$ is an affine maximizer if there is $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ such that

$$
\sum_{i=1}^{n} \sigma_id_{t_i}(f(t), a) + q(f(t), a) \geq 0
$$

for all $a \in A$ and all types $t \in T$.

Although Roberts (1979) asserts that truthful implementability and the affine maximization property are equivalent when $3 \leq |f(T)| < \infty$
and the compositions $T_i \rightarrow D(A) \rightarrow D(f(T))$ are all surjective, this is not quite correct.

**Example 2.4.** Suppose that $n = 2$ and $A$ is finite. Suppose that $f$ always chooses one of agent 1’s favorite alternatives, and if the set of favorites has more than one element, then $f$ chooses one of agent 2’s least favorite elements of this set. Clearly $f$ is an affine maximizer because it maximizes agent 1’s utility, but it is not truthfully implementable because for any payment scheme agent 2 is motivated to misreport some of her types.

As this example suggests, in order to obtain an exact characterization of truthful implementability we need to recursively impose conditions when the affine functional has more than one maximizer.

**Definition 2.5.** Suppose that $f$ is an affine maximizer, with $q$ and $\sigma$ as above, and let $S = \{ i : \sigma_i = 0 \}$, $T_S = \times_{i \in S} T_i$, and $T_{N \setminus S} = \times_{i \in N \setminus S} T_i$. We say that $f$ is a lexicographic affine maximizer if, for each $t_{N \setminus S} \in T_{N \setminus S}$:

(a) if $|f(t_{N \setminus S}, T_S)| = 2$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a binary implementable SCF;

(b) if $|f(t_{N \setminus S}, T_S)| \geq 3$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a lexicographic affine maximizer.

Of course the appearance of circularity in this definition is resolved by working inductively, assuming that we have already defined lexicographic affine maximization when there are fewer than $n$ agents.

We are concerned with finding circumstances under which truthful implementation and lexicographic affine maximization are equivalent. One of the two implications is straightforward, and does not require any additional assumptions.

**Proposition 2.** If $f$ is a lexicographic affine maximizer, then it is truthfully implementable in dominant strategies.

**Proof.** Let $\sigma \in \Delta^{n-1}$ and $q \in D(A)$ be as in Definition 2.3 with $S = \{ i : \sigma_i = 0 \}$. For each $i \in N \setminus S$ choose some $t^*_i \in T_i$, and let $p : T \rightarrow \mathbb{R}^n$ be the (generalized VCG) payment scheme given by

$$p_i(t) = \frac{1}{\sigma_i} \left\{ \sum_{j \in N \setminus S, j \neq i} \sigma_j d_{t_j}(f(t^*_i, t_{-i}), f(t)) + q(f(t^*_i, t_{-i}), f(t)) \right\}.$$ 

This construction guarantees that

$$d_{t_i}(f(t), f(t^*_i, t_{-i})) \geq p_i(t) - p_i(t^*_i, t_{-i});$$
for all \( t \in T, i \in N \setminus S \), and \( t_i' \in T_i \). For each \( t_{N \setminus S} \in \times_{i \in N \setminus S} T_i \) the derived SCF \( f(t_{N \setminus S}, \cdot) : \times_{i \in S} T_i \rightarrow A \) is trivially truthfully implementable if it is a constant function and automatically truthfully implementable if it is a binary implementable SCF. Otherwise it is a lexicographic affine maximizer. Therefore this argument can be repeated inductively to construct a payment scheme such that truth telling is incentive compatible for all \( i \in N \).

We now introduce another condition that \( f \) may satisfy. For \( w, z \in \mathbb{R}^n \) we write \( w \gg z \) to indicate that \( w_i > z_i \) for all \( i \).

**Definition 2.6.** We say that \( f \) is **monotonic in differences** if for any \( t, t' \in T \) there is some \( i \in N \) such that

\[
d_i(f(t), f(t')) \geq d_i(f(t), f(t')).
\]

In some contexts it is more direct to refer to the contrapositive, so we introduce distinct terminology for it: \( f \) is **negatively unanimous** if for all \( t, t' \in T \) and all \( a \in A \setminus \{f(t)\} \), if \( d_{t'}(f(t), a) \gg d_t(f(t), a) \), then \( f(t') \neq a \).

The critical issue is to find conditions that imply that \( T \) is “sufficiently rich” to insure that truthful implementability implies monotonicity in differences, which in turn implies affine maximization.

**Definition 2.7.** We say that \( T_i \) **allows elevation of pairs** if for all \( t_i, t_i' \in T_i \) and \( x, y \in A \) such that \( d_{t_i}(x, y) > d_{t_i}(x, y) \), there is a \( \hat{t}_i \in T_i \) such that

\[
d_{\hat{t}_i}(x, a) > d_{t_i}(x, a), \quad \text{for all } a \neq x,
\]

\[
d_{\hat{t}_i}(y, a) > d_{t_i}(y, a), \quad \text{for all } a \neq y.
\]

We say that \( T \) allows elevation of pairs if each \( T_i \) allows elevation of pairs.

**Proposition 3.** If \( T \) allows elevation of pairs and \( f \) is truthfully implementable, then \( f \) is monotonic in differences.

Combining Propositions 2 and 3, we see that truthful implementability implies monotonicity in differences when \( T \) allows elevation of pairs. The reverse implication holds without any qualifications.

**Proposition 4.** If \( f \) is an affine maximizer, then it is monotonic in differences.

**Proof.** Let \( \sigma \in \Delta^{n-1} \) and \( q \in D(A) \) be such that for every type \( t \in T \),

\[
\sum_{i=1}^{n} \sigma_i d_{t_i}(f(t), a) + q(f(t), a) \geq 0, \quad \text{for all } a \in A. \tag{1}
\]
To obtain a contradiction, suppose that monotonicity in differences fails for $f$. Then, there must exist a pair of types $t, t' \in T$ such that for every $i \in N$, 
\[ d_{t_i}(f(t), f(t')) < d_{t_i'}(f(t), f(t')). \]

Multiplying both sides of the above inequality by $\sigma_i$ and adding up across agents (recall $\sigma_i > 0$ for at least some $i \in N$), we see that 
\[ \sum_{i=1}^{n} \sigma_id_{t_i}(f(t), f(t')) < \sum_{i=1}^{n} \sigma_i d_{t_i'}(f(t), f(t')). \]

Since $f$ is an affine maximizer, expression (1) holds for $a = f(t')$, so that 
\[ \sum_{i} \sigma_id_{t_i}(f(t), f(t')) \geq -q(f(t), f(t')). \]

Using this together with the above equation, we obtain 
\[ 0 > -\sum_{i=1}^{n} \sigma_id_{t_i'}(f(t), f(t')) - q(f(t), f(t')) \]
\[ = \sum_{i=1}^{n} \sigma_i d_{t_i'}(f(t'), f(t)) + q(f(t'), f(t)), \]

which is a contradiction to $f$ being an affine maximizer. \[ \square \]

Allowing elevation of pairs is a condition that is imposed on each $T_i$ individually. In the following condition, on the other hand, the restriction is coordinated across the different $T_i$.

**Definition 2.8.** We say that $T$ is flexible if:

(F1) For any distinct $x, y \in A$ there are disjoint sets $B_x, B_y \subset A$ with $x \in B_x$ and $y \in B_y$ such that for all $i \in N$, $t_i^x, t_i^y \in T_i$, and $\delta_{xyi} \in \mathbb{R}$, there is a $t_i \in T_i$ satisfying 
\[ d_{t_i}(x, y) = \delta_{xyi}, \]

and further:

(a) $d_{t}(a, x) < d_{t'}(a, x)$, for all $a \in A \setminus \{x\} \cup B_y$;
(b) $d_{t}(a, y) < d_{t'}(a, y)$, for all $a \in A \setminus \{y\} \cup B_x$.

(F2) For any distinct $x, y, z \in A$ there are pairwise disjoint sets $B_x', B_y', B_z' \subset A$ with $x \in B_x'$, $y \in B_y'$, and $z \in B_z'$, such that for all $i \in N$, $t_i^x, t_i^y, t_i^z \in T_i$, and $\delta_{xyi}, \delta_{yzi} \in \mathbb{R}$, there is a $t_i \in T_i$ such that 
\[ d_{t_i}(x, y) = \delta_{xyi} \quad \text{and} \quad d_{t_i}(y, z) = \delta_{yzi}, \]

and further:

(a) $d_{t}(a, x) < d_{t'}(a, x)$, for all $a \in A \setminus \{x\} \cup B_y' \cup B_z'$;
(b) $d_{t}(a, y) < d_{t'}(a, y)$, for all $a \in A \setminus \{y\} \cup B_z' \cup B_x'$;
(c) $d_{t}(a, z) < d_{t'}(a, z)$, for all $a \in A \setminus \{z\} \cup B_y' \cup B_x'$. 
These properties abstract the key features of a topological setting that we will employ in Section 3. To develop intuition for them it may help to imagine that each profile \( d_i \) is continuous, and that the sets \( B_{x} \) and \( B'_{x} \) are certain neighborhoods of \( x \), and similarly for \( y \) and \( z \).

**Proposition 5.** If \( T \) is flexible, \( f \) is monotonic in differences, and \( |f(T)| \geq 3 \), then \( f \) is an affine maximizer.

We will prove Propositions 3 and 5 in Section 5. They combine with Proposition 2 to give our main result, which is the following characterization theorem.

**Theorem 1.** If \( T \) allows elevation of pairs and is flexible, then \( f \) is truthfully implementable if and only if one of the following hold:

1. \( f \) is a constant function;
2. \( f \) is a binary implementable SCF;
3. \( f \) is a lexicographic affine maximizer.

**Proof.** It is obvious that \( f \) is truthfully implementable if (a) holds, truthful implementability is part of the definition of (b), and we have already seen that (c) implies truthful implementability.

Suppose that \( f \) is truthfully implementable. Evidently there is nothing to prove unless \( |f(T)| \geq 3 \), in which case Propositions 3 and 5 imply that \( f \) is an affine maximizer, so there are \( g \) and \( \sigma \) such that the conditions of Definition 2.3 are satisfied. Let \( S = \{ i : \sigma_i = 0 \} \), \( T_S = \times_{i \in S} T_i \), and \( T_{N \setminus S} = \times_{i \in N \setminus S} T_i \), and consider \( t_{N \setminus S} \in T_{N \setminus S} \). Of course \( f(t_{N \setminus S}, \cdot) : T_S \rightarrow A \) is truthfully implementable. By induction we may assume that the result has already been established for SCF’s with fewer than \( n \) agents, so \( f(t_{N \setminus S}, \cdot) \) is constant, a binary implementable SCF, or a lexicographic affine maximizer. Since \( t_{N \setminus S} \) was arbitrary, the proof is complete.

\[ \square \]

### 3. Applications

In this section we explore some applications that the additional generality of our result opens up.

#### 3.1. Comprehensive Domains

A function \( \nu: A \rightarrow \mathbb{R} \) will be called a *valuation*. Let \( \mathcal{V}(A) \) be the space of all valuations. Our notation will usually not distinguish between a scalar in \( \mathbb{R} \) and the constant function with that value. Of course \( \mathcal{V}(A) \) is a vector space when endowed with the pointwise addition and scalar multiplication operations, and it is ordered using the
pointwise ordering. Given \( \nu, \nu' \in \mathcal{V}(A) \), we write \( \nu \lor \nu' \) and \( \nu \land \nu' \) for the pointwise max and min functions, respectively:

\[
\nu \lor \nu'(a) = \max\{\nu(a), \nu'(a)\} \quad \text{and} \quad \nu \land \nu'(a) = \min\{\nu(a), \nu'(a)\}.
\]

We let \( \nu^+ \) stand for \( \nu \lor 0 \) and \( |\nu| = \nu^+ + (-\nu)^+ \). The support of the valuation \( \nu \in \mathcal{V}(A) \) is the set

\[
\text{supp}(\nu) = \{ a \in A \mid \nu(a) \neq 0 \}.
\]

We assume that each agent \( i \in N \) is endowed with a set \( V_i \subseteq \mathcal{V}(A) \) of admissible valuations. Let

\[
V = V_1 \times \cdots \times V_n \subseteq \mathcal{V}(A)^n
\]

We also assume that each \( V_i \) is closed under addition of constants, so that \( v_i + \delta_i \in V_i \) whenever \( v_i \in V_i \) and \( \delta_i \in \mathbb{R} \).

Given \( \nu \in \mathcal{V}(A) \), there is an induced difference \( d_\nu \) defined by

\[
d_\nu(a, b) = \nu(a) - \nu(b).
\]

In this sense, \( V \) may be regarded as a space of type profiles of the sort considered in the preceding section. Our goal is to develop conditions on \( V \) that imply that it is flexible and allows elevation of pairs.

**Definition 3.1.** A set \( \mathcal{U} \) of functions from \( A \) to \([0, 1]\) is called a separating family if, for any distinct alternatives \( x, y, z \in A \), there are functions \( \mu^x, \mu^y, \mu^z \in \mathcal{U} \) satisfying \( \mu^x(x) = \mu^y(y) = \mu^z(z) = 1 \), whose supports are pairwise disjoint:

\[
\text{supp}(\mu^x) \cap \text{supp}(\mu^y) = \text{supp}(\mu^x) \cap \text{supp}(\mu^z) = \text{supp}(\mu^y) \cap \text{supp}(\mu^z) = \emptyset.
\]

For example, if the set of alternatives \( A \) is a metric space with metric \( d \), then there is a separating family

\[
\{ \mu^{xs} \mid x \in A \text{ and } s > 0 \} \quad \text{where} \quad \mu^{xs}(a) = \max\{1 - \frac{1}{s}d(x, a), 0\}.
\]

Fix a separating family \( \mathcal{U} \subseteq \mathcal{V}(A) \). The functions in \( \mathcal{U} \) need not be admissible valuations themselves, but will be used to perturb admissible valuations.

**Definition 3.2.** The domain \( V \) is \( \mathcal{U} \)-comprehensive if:

(C1) For all \( i \in N, v_i \in V_i, \mu \in \mathcal{U}, x \in A \), and \( \delta_i \in \mathbb{R}_+ \), there is a valuation \( \hat{v}_i \in V_i \) such that

\[
\hat{v}_i(x) = v_i(x) + \delta_i \mu(x).
\]
and
\[ \hat{v}_i(a) < v_i(a) + \delta_i \mu(a) \]
for all \( a \in A \setminus \{x\} \).

(C2) For all \( i \in N \), \( v_i, v'_i \in V_i \), and \( x, y \in A \), there exists \( \hat{v}_i \in V_i \) such that
\[ \hat{v}_i(a) \leq v_i(a) \]
for all \( a \in A \), with equality when \( a = x \) and when \( a = y \).

(C3) For all \( i \in N \), \( v_i, v'_i, v''_i \in V_i \), and \( x, y, z \in A \), there is a \( \hat{v}_i \in V_i \) such that \( \hat{v}_i(x) = v_i(x) \), \( \hat{v}_i(y) = v'_i(y) \), \( \hat{v}_i(z) = v''_i(z) \), and
\[ \hat{v}_i(a) \leq v_i \lor v'_i \lor v''_i(a) \]
for all \( a \in A \).

**Lemma 3.3.** If \( V \) is \( U \)-comprehensive, then for any set \( K \subset A \) with either two or three elements and any valuation profile \( v^x \in V \) indexed for the various \( x \in K \), there exists an admissible profile \( v \in V \) such that \( v \leq \bigwedge_{x \in K} v^x \).

**Proof.** First suppose that \( K = \{x, y, z\} \). Applying (C2) to \( v^x_i \) and \( v^y_i \), we see the existence of some \( \hat{v}_i \in V_i \) such that \( \hat{v}_i(a) \leq v^x_i(a) \lor v^y_i(a) \) for all \( a \in A \). Repeating this argument, for each \( i \in N \) choose \( v_i \leq \hat{v}_i \lor v^x_i \) in \( V_i \). The profile \( v = (v_1, \ldots, v_n) \) is our desired valuation profile.

When \( K \) has only two alternatives, we simply terminate the construction after the first step, setting \( v_i = \hat{v}_i \).

As usual for \( \delta, \gamma \in \mathbb{R}^n \), we write \( \delta \ll \gamma \) when \( \delta_i < \gamma_i \) for all \( i \in N \).

**Lemma 3.4.** Suppose that \( V \) is \( U \)-comprehensive and \( K \subset A \) has two or three alternatives. Then for any valuation profiles \( v^x \) indexed for the various \( x \in K \), there are pairwise disjoint sets \( B_x \subset A \), with \( x \in B_x \) for each alternative \( x \in K \), and an admissible profile \( v \in V \) such that
\[ v(a) \ll \bigwedge_{x \in K} v^x(a), \quad \text{for all } a \in A \setminus \bigcup_{x \in K} B_x, \]
and further, for each \( x \in K \), \( v(x) = v^x(x) \) and \( v(a) \ll v^x(a) \) for all \( a \in B_x \setminus \{x\} \).

**Proof.** Fix valuations \( v^x \in V \) for the various \( x \in K \). Let \( \{\mu^x \mid x \in K\} \) be a subset of the separating family \( U \), so that \( \mu^x(x) = 1 \) for each \( x \in K \) and the supports \( B_x = \text{supp}(\mu^x) \) being pairwise disjoint. Applying the result above, choose \( \tilde{v} \in V \) such that
\[ \tilde{v} \leq \bigwedge_{x \in K} v^x. \]
For each $i \in N$, choose $\delta_i > 0$ large enough that $\tilde{v}_i(x) + \delta_i > v_i^x(x)$ for every $x$ in $K$. For any particular $x$, (C1) applied to $\tilde{v}_i$, $\mu^x$ and $\delta_i$ implies that there is a $\hat{v}_i^x \in V_i$ such that $\hat{v}_i^x(x) = \tilde{v}_i(x) + \delta_i$ and $\hat{v}_i^x(a) < \tilde{v}_i(a) + \delta_i \mu^x(a)$ for all $a \in A$, $a \neq x$. Applying (C2) to $v_i^x$, $\hat{v}_i^x \in V_i$, we conclude that there is a $\hat{v}_i^x \in V_i$ such that $\hat{v}_i^x(x) = v_i^x(x)$ and $\hat{v}_i^x(a) \leq \hat{v}_i^x \wedge v_i^x(a)$, for all $a \in A \setminus \{x\}$.

Consider $a \in B_x \setminus \{x\}$. For $y \in K$, $y \neq x$, we have $\mu^y(a) = 0$ and thus

$$\tilde{v}_i^y(a) \leq \hat{v}_i^y(a) < \tilde{v}_i(a) \leq \bigwedge_{y \in K} \tilde{v}_i(y).$$

Since $\tilde{v}_i^y(a) \leq v_i^y(a)$, we conclude that $\tilde{v}_i^y(a) \leq v_i^y(a)$ for all $y \in K$ and all $a \in B_x \setminus \{x\}$. Note as well that for $a \in A \setminus \bigcup_{x \in K} B_x$, it follows that $\tilde{v}_i^y(a) < \tilde{v}_i(a)$ for each $x \in K$. Since all this holds for any $i \in N$, we can conclude that for all $x, y \in K$,

$$\bar{v}_x^y(a) \leq v_y(a)$$

for all $a \in A \setminus K$.

By (C1) with $\delta_i = 0$, for each $x \in K$ there is a profile $\bar{v}_x \in V$ such that $\bar{v}_x(x) = \tilde{v}_x(x)$ and $\bar{v}_x(a) \ll \tilde{v}_x(a)$, for all $a \neq x$. Finally, applying (C3) to $\bar{v}_x$, all $x \in K$, we deduce the existence of an admissible profile $v \in V$ such that

$$v(x) = \bar{v}_x(x) = \tilde{v}_x(x) = v^x(x)$$

for all $x \in K$,

$$v(a) \leq \bigvee_{x \in K} \tilde{v}_x(a) \ll v^x(a)$$

for all $a \in B_x \setminus \{x\}$; and

$$v(a) \leq \bigvee_{x \in K} \tilde{v}_x(a) \ll \bigwedge_{x \in K} v^x(a)$$

for all $a \in A \setminus \bigcup_{x \in K} B_x$. \hfill \Box

In light of Lemma 3.4, it is easily established that flexibility follows from $\mathcal{U}$-comprehensiveness.

**Corollary 3.5.** If $V$ is $\mathcal{U}$-comprehensive, then it is flexible.

**Proof.** We will only prove (F1); the proof of (F2) follows the same pattern. Fix distinct $x, y \in A$. Choose $\mu^x, \mu^y \in \mathcal{U}$ such that $\mu^x(x) = \mu^y(y) = 1$ and $B_x = \text{supp}(\mu^x)$ and $B_y = \text{supp}(\mu^y)$ are disjoint. Because the space of profiles is a cartesian product, it now suffices to prove that for any $v^x, v^y \in V$ and $\delta_{xy} \in \mathbb{R}^n$, there is a profile $v \in V$ such that
Corollary 3.7. Suppose that $d_{v}(x, y) = \delta_{xy}$, $d_{v}(a, x) \ll d_{v}(a, x)$ for all $a \in A \setminus \{x\} \cup B_{y}$, and $d_{v}(a, y) \ll d_{v}(a, y)$ for all $a \in A \setminus \{y\} \cup B_{x}$.

Fix $v^{x}, v^{y} \in V$ and $\delta_{xy} \in \mathbb{R}^{n}$. Choose $\delta_{x}, \delta_{y} \in \mathbb{R}^{n}$ such that

$$v^{x}(x) - v^{y}(y) + \delta_{x} - \delta_{y} = \delta_{xy}.$$ 

By (C1) there are $\hat{v}^{x}, \hat{v}^{y} \in V$ such that $\hat{v}^{x}(x) = v^{x}(x) + \delta_{x}$, $\hat{v}^{y}(y) = v^{y}(y) + \delta_{y}$, $\hat{v}_{x}(a) < v_{x}(a) + \mu^{x}(a)\delta_{x}$ for all $a \in A \setminus \{x\}$, and $\hat{v}_{y}(a) < v_{y}(a) + \mu^{y}(a)\delta_{y}$ for all $a \in A \setminus \{y\}$. Lemma 3.4 now gives a $v \in V$ such that $v(x) = \hat{v}^{x}(x) = v^{x}(x) + \delta_{x}$, $v(y) = \hat{v}^{y}(y) = v^{y}(y) + \delta_{y}$,

$$v(a) \ll \hat{v}^{x} \wedge \hat{v}^{y}(a)$$

for all $a \in A \setminus (B_{x} \cup B_{y})$,

$$v(a) \ll \hat{v}^{x}(a)$$

for all $a \in B_{x} \setminus \{x\}$, and

$$v(a) \ll \hat{v}^{y}(a)$$

for all $a \in B_{y} \setminus \{y\}$.

If $a \in A \setminus (B_{x} \cup B_{y})$, then

$$v(a) \ll \hat{v}^{x}(a) \ll v^{x} \wedge v^{y}(a).$$

If $a \in B_{x} \setminus \{x\}$, then

$$v(a) \ll \hat{v}^{x}(a) \ll v^{x}(a) + \mu^{x}(a)\delta_{x}.$$

Of course $v(x) = v^{x}(x) + \delta_{x}$, so in either case one has

$$d_{v}(a, x) \ll d_{v^{x}}(a, x).$$

Reversing $x$ and $y$ in this argument, we find that $d_{v}(a, y) \ll d_{v^{y}}(a, y)$ for all $a \in A \setminus \{y\} \cup B_{x}$. $\square$

Lemma 3.6. Suppose that $V$ is $U$-comprehensive. If $v, v' \in V$ and $x, y \in A$ satisfy $d_{v}(x, y) \gg d_{v}(x, y)$, then there exists $\hat{v} \in V$ such that $d_{\hat{v}}(x, a) \gg d_{v}(x, a)$ for all $a \neq x$ and $d_{\hat{v}}(y, a) \gg d_{v}(y, a)$ for all $a \neq y$.

Proof. We can apply (C2) to $v$ and $v'$, at $x$ and $y$ respectively, to obtain a profile $\bar{v}$ in $V$ for which $\bar{v}(x) = v(x)$, $\bar{v}(y) = v'(y)$, and $\bar{v}(a) \leq v \land v'(a)$, all $a \in A, a \neq x, y$. Now (C1) implies the existence of $v^{x}, v^{y} \in V$ such that $v^{x}(x) = \bar{v}(x) = v(x)$ and $v^{x}(a) \ll \bar{v}(a)$, for all $a \neq x$, and similarly $v^{y}(y) = \bar{v}(y) = v'(y)$ and $v^{y}(a) \ll \bar{v}(a)$, for all $a \neq y$. By Lemma 3.4, there exists an admissible profile $\hat{v} \in V$ such that $\hat{v}(x) = v^{x}(x) = \bar{v}(x)$, $\hat{v}(y) = v^{y}(y) = v'(y)$, and $\hat{v}(a) \ll v^{x} \lor v^{y}(a)$, for each $a \neq x, y$. Evidently $\hat{v}$ satisfies the desired conditions. $\square$

Corollary 3.7. If $V$ is $U$-comprehensive, then it allows elevation of pairs.
Proof. Since $V$ is a cartesian product, this follows from Lemma 3.6. \hfill\Box

The usefulness of a comprehensive domain of admissible valuations will be displayed next.

### 3.2. Resolving Lattices

In certain economic applications it may be reasonable to model the set of alternatives as being infinite and the domain of admissible valuation profiles as containing only continuous, piecewise linear, or even differentiable functions. We show here how Theorem 1 can be employed in these cases. We first prove that our main result holds true for the important case where $V$ is a vector lattice subspace of $V(A)^n$ which includes constant functions and resolving functions. It follows that continuous domains are comprehensive. In the Appendix we show that under certain restrictions imposed on the set of alternatives, Theorem 1 remains valid when the domain of admissible valuation profiles contains only smooth functions defined on $A$.

**Definition 3.8.** A subset $L$ of $V(A)$ is called a *resolving set* if for every alternative $x \in A$ there exists a function $\nu^x \in L$ such that $\nu^x(a) \neq \nu^x(x)$, for all $a \in A$, $a \neq x$. The function $\nu^x$ is said to be a resolving function of $x$.

An enriched class of resolving sets is defined next.

**Definition 3.9.** A subset $\mathcal{L}(A)$ of $V(A)$ is called a *resolving lattice* if the following are satisfied:

1. $\mathcal{L}(A)$ is a linear subspace containing the constant function 1;
2. $\mathcal{L}(A)$ is a resolving set;
3. $\mathcal{L}(A)$ is a lattice; i.e., if $\nu, \nu' \in \mathcal{L}(A)$, then both functions $\nu \vee \nu'$ and $\nu \wedge \nu'$ belong to $\mathcal{L}(A)$.

Notice that if $\mathcal{L}(A) \subseteq V(A)$ is a resolving lattice, then for any finite subset $K$ of $A$, the space of restricted functions $\{\nu|_K \mid \nu \in \mathcal{L}(A)\}$ coincides with $\mathbb{R}^{|K|}$.

**Lemma 3.10.** If $\mathcal{L}(A)$ is a resolving lattice, then it contains a separating family.

Proof. Let $x, y, z$ be distinct alternatives in $A$. By taking a linear combination of a function with different values at $x$ and $y$ and a function with different values at $y$ and $z$, we can construct an element of $\mathcal{L}(A)$ taking different values at the three points. After permuting $x$, $y$, and $z$, and adding a suitable constant, we obtain $\nu \in \mathcal{L}(A)$ with $\nu(x) > 0 = \nu(y) > \nu(z)$. Since we can replace $\nu$ with $s\nu^x + t\nu$ for
suitable \( s, t > 0 \), we can in fact assume that \( \nu(x) = 2, \nu(y) = 0, \) and \( \nu(z) = -2 \). Let
\[
\mu^x = (2 \wedge \nu - 1)^+, \quad \mu^y = 1 - (1 \wedge |\nu|), \quad \mu^z = (2 \wedge (-\nu) - 1)^+.
\]
Clearly \( \mu^x, \mu^y, \mu^z \in \mathcal{L}(A) \). It is easily checked that the images of these functions are contained in \([0, 1]\) with \( \mu^x(x) = \mu^y(y) = \mu^z(z) = 1 \), and that their supports are pairwise disjoint.

**Proposition 6.** If \( \mathcal{L}(A) \subseteq \mathcal{V}(A) \) is a resolving lattice, then \( V = \mathcal{L}(A)^n \) is \( \mathcal{L}(A) \)-comprehensive.

**Proof.** (C1) Consider \( i \in \mathbb{N}, v_i \in V_i = \mathcal{L}(A), \mu \in \mathcal{L}(A) \) mapping \( A \) to \([0, 1]\), \( x \in A \), and \( \delta_i \in \mathbb{R}_+ \). We know there is \( \nu^x \in \mathcal{L}(A) \) such that \( \nu^x(x) \neq \nu^x(a) \) for all \( a \in A, a \neq x \). Let \( \nu = |\nu^x - \nu^x(x)| \). Then \( i \)'s valuation
\[
\hat{v}_i = v_i + \delta_i \mu - \nu
\]
is in \( V_i \) and has the desired properties.

(C2) This condition holds simply because \( V_i = \mathcal{L}(A) \) is a lattice.

(C3) Suppose that \( v_i, v'_i, v''_i \in V_i = \mathcal{L}(A) \) and that \( x, y, z \in A \) are distinct. The last result gives functions \( \mu^x, \mu^y, \mu^z \in \mathcal{L}(A) \) taking values in \([0, 1]\) with pairwise disjoint supports and \( \mu^x(x) = \mu^y(y) = \mu^z(z) = 1 \). If \( \delta_i \geq 0 \) is sufficiently large, then
\[
\hat{v}_i = [v_i + \delta_i(\mu^y + \mu^z)] \wedge [v'_i + \delta_i(\mu^x + \mu^y)] \wedge [v''_i + \delta_i(\mu^x + \mu^y)]
\]
belongs to \( V_i \) and has the desired properties. \( \square \)

**Example 3.11.** Let \( \mathcal{L} \) be an arbitrary resolving subset of \( \mathcal{V}(A) \). Let \( R(\mathcal{L}) \) be the space of all functions of the form
\[
\max_{j=1}^k \min_{j'=1}^{k'} \alpha_{jj'} + \nu_{jj'},
\]
where \( k \) and \( k' \) are two natural numbers, each \( \alpha_{jj'} \) is a real number, and each \( \nu_{jj'} \) is a linear combination of functions in \( \mathcal{L} \). Notice that the space \( R(\mathcal{L}) \) is a resolving lattice. Hence, for \( V + \mathbb{R}^n = R(\mathcal{L})^n \), the domain \( V \) is comprehensive.

Proposition 6 shows that one can generate infinitely many distinct examples of comprehensive domains. When \( A \) is a topological space, we let \( \mathcal{C}(A) \) be the space of all continuous functions on \( A \).

**Corollary 3.12.** If \( A \) is a topological space such that for each \( x \in A \), there exists a continuous resolving function \( \nu^x \) of \( x \), then \( V = \mathcal{C}(A)^n \) is \( \mathcal{C}(A) \)-comprehensive.
Suppose now that $A$ is a convex subset of a finite dimensional Euclidean space $\mathbb{R}^m$. A function $\nu: A \rightarrow \mathbb{R}$ is said to be piecewise affine if there is a finite number $\nu^1, \ldots, \nu^k$ of affine functions from $A$ to $\mathbb{R}$ such that for each $a \in A$ we have $\nu(a) = \nu^j(a)$ for some $j = 1, \ldots, k$. Let $\mathcal{P}(A)$ be the space of continuous piecewise affine functions on $A$.

**Corollary 3.13.** If $A$ is a convex subset of $\mathbb{R}^m$, for some positive integer $m$, then $V = \mathcal{P}(A)^n$ is comprehensive.

**4. Examples**

In this section we discuss examples that clarify the extent to which the various implications that constitute Theorem 1 hold in domains that are not comprehensive. The first example, inspired by Example S3 in Bikhchandani, Chatterji, Lavi, Mu’alem, Nisan, and Sen (2006a), illustrates this point.

**Example 4.1.** There are three units of a good to be allocated among three agents with unit demands. We use $o_i = y$ to indicate that agent $i \in N = \{1, 2, 3\}$ receives one unit of the good, and $o_i = n$ to indicate otherwise. Thus, an allocation is represented by $o_1o_2o_3$, and the allocation set is $A = \{o_1o_2o_3 \mid o_i = y, n, i \in N\}$. Agent $i'$'s valuation domain $V_i$ contains all vectors $v_i \in \mathbb{R}^{||A||}$ satisfying:

$$v_i(o_1o_2o_3) = \begin{cases} 
\alpha_i, & \text{if } o_i = y, \\
0, & \text{if } o_i = n;
\end{cases}$$

where $\alpha_i \in \mathbb{R}$ for each $i = 1, 2, 3$.

Consider first the social choice function $f: V \rightarrow A$ that selects $f(v) = o_1o_2o_3$, with $o_i = y$ if and only if $\alpha_i > \alpha_j + \alpha_k - 10$, where $j$ and $k$ denote the two other agents. It is immediate to check that $f$ is truthfully implemented by the payment rule $p: V \rightarrow A$ defined, for all $i \in N$, by $p_i(v) = \alpha_j + \alpha_k - 10$ if $f(v)$ satisfies $o_i = y$ and $p_i(v) = 0$ otherwise. However, $f$ is not monotonic in differences. To see this, let $v, v' \in V$ be two valuation profiles such that for each $i$, $\alpha_i = 9$ and $\alpha_i' = 11$. Clearly, $f(v) = yyy$ whereas $f(v') = nnn$. Notice now that for all $i \in N$, $v'_i(f(v')) - v'_i(f(v)) = -11$ and $v_i(f(v')) - v_i(f(v)) = -9$, which is a violation of monotonicity in differences. Thus Proposition 3 does not hold without the assumption of elevation of pairs.

Consider instead $\hat{f}: V \rightarrow A$ defined by $\hat{f}(v) = o_1o_2o_3$, where for $i = 1, 2$, $o_i = y$ if and only if $\alpha_i < \alpha_j + \alpha_k - 10$, and $o_3 = y$ if and only if $\alpha_3 > 0$. To show that monotonicity in differences is satisfied, fix two admissible profiles $v, v' \in V$ and let $\hat{f}(v) = o_1o_2o_3$ and $\hat{f}(v') = o'_1o'_2o'_3$. If it is the case that $o_3 = o'_3$, then immediately one has
$v'_3(\hat{f}(v')) - v'_3(\hat{f}(v)) = v_3(\hat{f}(v')) - v_3(\hat{f}(v))$. Suppose instead that $o_3 = n$ and $o'_3 = y$, hence $\alpha_3 \leq 0 < \alpha'_3$. Then, since $v'_3(\hat{f}(v')) - v'_3(\hat{f}(v)) = \alpha'_3$ whereas $v_3(\hat{f}(v')) - v_3(\hat{f}(v)) = \alpha_3$, it follows $\hat{f}$ satisfies monotonicity in differences. We now claim that $\hat{f}$ is not implementable. Indeed, fix $v_2, v_3$ and let $v_1, v'_1$ be such that $\alpha_1 \geq \alpha_2 + \alpha_3 - 10 > \alpha'_1$, hence $\hat{f}(v_1, v_{-1}) = no_2 o_3$ and $\hat{f}(v'_1, v_{-1}) = yo_2 o_3$. Notice that $v'_1(\hat{f}(v'_1, v_{-1})) - v'_1(\hat{f}(v_1, v_{-1})) = \alpha'_1 < \alpha_1 = v_1(\hat{f}(v'_1, v_{-1})) - v_1(\hat{f}(v_1, v_{-1}))$, which violates the standard weak monotonicity condition (Bikhchandani, Chatterji, Lavi, Mu’alem, Nisan, and Sen (2006a)). Since weak monotonicity is necessary for implementation in any environment, this shows that $\hat{f}$ is not truthfully implementable in dominant strategies. In this case monotonicity in differences does not imply truthfully implementability.

Finally, we observe that $\hat{f}$ is not an affine maximizer. Suppose to obtain a contradiction that there are weights $(\sigma_1, \sigma_2, \sigma_3) \in \Delta^3$ and a (finite) collection of real numbers $\{q(o_1 o_2 o_3): o_1 o_2 o_3 \in A\}$ such that $\hat{f}(v) \in \arg \max \{\sum^3_1 \sigma_i v_i(o_1 o_2 o_3) + q(o_1 o_2 o_3): o_1 o_2 o_3 \in A\}$ for all $v \in V$. Let $v'$ be such that $\hat{f}(v'_1, v'_{-1}) = yo'_2 o'_3$, and consider a profile $(v''_1, v''_{-1})$ satisfying $\alpha''_1 > \alpha'_1$. It follows that $\hat{f}(v''_1, v''_{-1}) = yo'_2 o'_3$ since the choice function $\hat{f}$ is an affine maximizer. However, from the definition of $\hat{f}$, one sees that $\hat{f}(v''_1, v''_{-1}) = no'_2 o'_3$ for $\alpha''_1$ sufficiently large, which is a contradiction. Thus Proposition 5 requires the assumption that the domain is flexible.

Example 4.1 shows that truthful implementation and monotonicity in differences are not equivalent conditions in every domain $V$. It also illustrates that the affine maximization property of a choice function does not necessarily follow from monotonicity in differences. Neither does affine maximization follow from truthful implementation; Mishra and Sen (2010) present an example showing this, which they attribute to Meyer-ter-Vehn and Moldovanu\(^5\). The following example also illustrates this point.

**Example 4.2.** The alternative set is $A = \{a, b, c, d\}$. Two agents have partially ordered preferences over $A$. In particular, $b \succeq_1 c, d \succeq_1 a$, and $c, d \succeq_2 b \succeq_2 a$. We consider valuation profiles that are consistent with these initial partial orders. A typical element of $V_i$ is $v_i = (v_i(a), v_i(b), v_i(c), v_i(d))$. The domain of valuation profiles $V = V_1 \times V_2$

\(^5\)Private communication.
is defined by:
\[ V_1 = \{ v_1 \in \mathbb{R}^4 \mid 0 < v_1(a) \leq v_1(c), v_1(d) \leq v_1(b) \} , \]
\[ V_2 = \{ v_2 \in \mathbb{R}^4 \mid 0 < v_2(a) \leq v_2(b) \leq v_2(c), v_2(d) \} . \]

Notice that the domain \( V \) is open, convex (hence connected), and unbounded from above. It however not comprehensive. Consider now the following subsets of \( V_1, V_2 \):
\[ V_1^\circ = \{ v_1 \in V_1 \mid 0 < v_1(a) < \frac{1}{2}, v_1(c) = v_1(d) = v_1(a), v_1(b) = v_1(a) + \frac{1}{2} \} , \]
\[ V_2^\circ = \{ v_2 \in V_2 \mid 0 < v_2(a) < \frac{1}{2}, v_2(b) = v_2(a), v_2(c) = v_2(d) = v_2(a) + \frac{1}{2} \} . \]

We define the social choice function \( f : V \to A \) by \( f(v) = a \) if and only if \( v_1 \in V_1^\circ \) and \( v_2 \in V_2^\circ \), and \( f(v) = \arg \max \{ v_1(k) + v_2(k) \mid k \in A, k \neq a \} \) otherwise. Notice that \( f \) is not an affine maximizer; it is however truthfully implementable in dominant strategy.

Indeed, using the Taxation Principle, we write the payment function \( p_i \) in terms of \( v_i \) and chosen alternative \( k \in A \). Consider the payment scheme given by:
\[
p_1(k, v_2) = \begin{cases} v_2(a), & \text{for } v_2 \in V_2^\circ, k = a, c, d, \\ v_2(a) + \frac{1}{2}, & \text{for } v_2 \in V_2^\circ, k = b, \\ -v_2(k), & \text{for } v_2 \in V_2 \setminus V_2^\circ, k \in A; \end{cases}
\]
and similarly
\[
p_2(k, v_1) = \begin{cases} v_1(a), & \text{for } v_1 \in V_1^\circ, k = a, b, \\ v_1(a) + \frac{1}{2}, & \text{for } v_1 \in V_1^\circ, k = c, d, \\ -v_1(k), & \text{for } v_1 \in V_1 \setminus V_1^\circ, k \in A. \end{cases}
\]

The reader can verify without difficulty that this payment scheme truthfully implements \( f \) in dominant strategies.

The domain \( V_i \) in Example 4.2 is an open and convex subset of \( \mathbb{R}^{\vert A \vert} \) but fails to be comprehensive. On the other hand, it is easy to construct examples of comprehensive domains \( V \) that are nonconvex subsets of \( \mathbb{R}^{\vert A \vert \times n} \).

**Example 4.3.** Let \( A \) be finite (with at least four alternatives). For any \( v_i \in V_i \) and any \( K \subset A \) with three elements, define
\[ V_i(v_i, K) = \{ \hat{v}_i \in V_i \mid \hat{v}_i|_A \setminus K = v_i|_A \setminus K \} . \]
That is, \( V_i(v_i, K) \subset V_i \) is composed of all of agent \( i \)'s valuations that agree with \( v_i \) except possible at \( x \in K \). Clearly, if for all \( i \in N, v_i \in V_i \), and \( K = \{ x, y, z \} \subset A \), it is the case that
\[ \{ \hat{v}_i|_K \mid \hat{v}_i \in V_i(v_i, K) \} = \mathbb{R}^3 , \]
then \( V \) is comprehensive. Observe \( V \) need not be a linear space nor a dense set in \((\mathbb{R}^A)^n\).

5. Proofs of Propositions 3 and 5

The first of the two proofs in this section is relatively straightforward.

Proof of Proposition 3. By hypothesis there is a payment scheme \( p: T \to \mathbb{R}^n \) such that for all \( i \in N, t \in T, \) and \( t'_i \in T_i, \)
\[
d_{t_i}(f(t), f(t'_i, t_{-i})) \geq p_i(t) - p_i(t'_i, t_{-i}).
\]
Aiming at a contradiction, suppose that \( f \) is not monotonic in differences, so there is a failure of negative unanimity: for some \( t, t' \in T \) we have
\[
d_{t'}(f(t), f(t')) \gg d_t(f(t), f(t')).
\]
Letting \( f(t) \) and \( f(t') \) take the role of \( x \) and \( y \), respectively, since each \( T_i \) allows elevation of pairs there is an admissible valuation profile \( \tilde{t} \in T \) such that
\[
d_{\tilde{t}}(f(t), a) > d_{\tilde{t}}(f(t), a), \quad \text{all } a \neq f(t), \quad \text{(2)}
\]
\[
d_{\tilde{t}}(f(t'), a) > d_{t'}(f(t'), a), \quad \text{all } a \neq f(t'). \quad \text{(3)}
\]
Let \( \hat{t}^0 = t \), and define \( \hat{t}^1, \ldots, \hat{t}^n \) inductively by setting \( \hat{t}^i = (\hat{t}^i, \hat{t}^{i-1} - 1) \). Truthful implementation implies that for each \( i \in N \) we have both
\[
d_{\hat{t}^i}(f(\hat{t}^i), f(\hat{t}^{i-1})) \geq p_i(\hat{t}^i) - p_i(\hat{t}^{i-1})
\]
and
\[
d_{\hat{t}^{i-1}}(f(\hat{t}^{i-1}), f(\hat{t}^i)) \geq p_i(\hat{t}^{i-1}) - p_i(\hat{t}^i).
\]
Combining these, and recognizing that \( \hat{t}^{i-1} = t_i \) and \( \hat{t}^i = \hat{t}_i \), gives
\[
d_x(f(t^{i-1}), f(\hat{t}^i)) \geq d_{t_i}(f(\hat{t}^{i-1}), f(\hat{t}^i)).
\]
Since \( \hat{t}^0 = t \) we have \( f(\hat{t}^0) = f(t) \), and if \( i \) is the first index such that \( f(\hat{t}^i) \neq f(t) \), this inequality contradicts (2). Therefore we have
\[
f(t) = f(\hat{t}^0) = \cdots = f(\hat{t}^n) = f(\hat{t}).
\]
Repeating this argument gives \( f(t') = f(\hat{t}) \), but \( f(t) = f(t') \) is impossible. \( \square \)

It remains to prove Proposition 5. For the remainder of the section we assume that \( T \) is flexible, and we fix a surjective social choice function \( f: T \to A \) with \( |f(T)| \geq 3 \) satisfying monotonicity in differences. For any distinct \( x, y \in A \), let \( Q(x, y) = d_t(x = f(t), y) + \mathbb{R}^n_{++} \). That is,
\[
Q(x, y) = \{ \alpha \in \mathbb{R}^n \mid d_t(x, y) \ll \alpha \text{ for some } t \in T \text{ with } f(t) = x \}.
\]
Notice that \( Q(x, y) \) is nonempty since the SCF \( f \) is surjective.

**Lemma 5.1.** For any distinct \( x, y \in A \), \( Q(x, y) \cap -Q(y, x) = \emptyset \).

**Proof.** Aiming at a contradiction, suppose that \( \alpha \in Q(x, y) \) and \( -\alpha \in Q(y, x) \). Choose \( t \in T \) such that \( d_t(x, y) \ll \alpha \) and \( f(t) = x \), and choose \( t' \in T \) satisfying \( d_{t'}(y, x) \ll -\alpha \) and \( f(t') = y \). But \( d_{t'}(y, x) = -d_t(x, y) \), so we have

\[
d_t(x, y) \gg \alpha \gg d_{t'}(x, y),
\]

which is a violation of negative unanimity. \( \square \)

The next two results develop the consequences of (F1).

**Lemma 5.2.** For any distinct \( x, y \in A \) and any \( \delta_{xy} \in \mathbb{R}^n \) there exists an admissible profile \( t \in T \) such that \( d_t(x, y) = \delta_{xy} \) and \( f(t) \in \{x, y\} \).

**Proof.** Since \( f \) is surjective there are \( t^x, t^y \in T \) such that \( f(t^x) = x \) and \( f(t^y) = y \). Let \( B_x \) and \( B_y \) be as in (F1). Then (F1) provides a \( t \in T \) such that \( d_t(x, y) = \delta_{xy} \) and:

(a) \( d_t(a, x) \ll d_t(a, y) \) for all \( a \in A \setminus \{x\} \cup B_y \);
(b) \( d_t(a, y) \ll d_{t^y}(a, y) \) for all \( a \in A \setminus \{y\} \cup B_x \).

Since \( (A \setminus \{x\} \cup B_y) \cup (A \setminus \{x\} \cup B_y) = A \setminus \{x, y\} \), if \( f(t) \notin \{x, y\} \) then either (a) or (b) gives a failure of negative unanimity. \( \square \)

**Lemma 5.3.** For any distinct \( x, y \in A \) and \( \alpha, \beta \in \mathbb{R}^n \) with \( \alpha \gg \beta \), either \( \alpha \in Q(x, y) \) or \( \beta \in -Q(y, x) \).

**Proof.** Lemma 5.2 gives a \( t \in T \) such that \( d_t(x, y) = \alpha \) and \( f(t) \in \{x, y\} \). That is, either \( f(t) = x \) and thus \( \alpha \in Q(x, y) \), or \( f(t) = y \), in which case \( -\beta \in Q(y, x) \) because \( d_t(y, x) + (\alpha - \beta) = -\beta \) and \( \alpha - \beta > 0 \). \( \square \)

Let \( e = (1, \ldots, 1) \in \mathbb{R}^n \). For each distinct \( x, y \in A \), let

\[
q(x, y) = \sup \{ t \in \mathbb{R} \mid -te \in Q(x, y) \}.
\]

**Lemma 5.4.** For any distinct \( x, y \in A \), \( q(x, y) = -q(y, x) \) and

\[
-q(x, y)e \notin (Q(x, y) \cup -Q(y, x))
\]

**Proof.** In view of Lemma 5.1, \( -q(x, y)e \ll q(y, x)e \) is impossible, so \( -q(x, y) \geq q(y, x) \). The inequality cannot be strict because then \( -\frac{1}{3}q(x, y) + \frac{2}{3}q(y, x)e \) and \( -\frac{2}{3}q(x, y) + \frac{1}{3}q(y, x)e \) would be in neither \( Q(x, y) \) nor \( -Q(y, x) \), which would contradict Lemma 5.3. In view of the definition of \( Q(x, y) \) it is obvious that \( -q(x, y)e \notin Q(x, y) \) and \( -q(y, x)e \notin Q(y, x) \). \( \square \)

The next three results develop the consequences of (F2).
Lemma 5.5. For any distinct $x, y, z \in A$ and any $\delta_{xy}, \delta_{yz} \in \mathbb{R}^n$ there exists an admissible profile $t \in T$ such that $d_t(x, y) = \delta_{xy}, d_t(y, z) = \delta_{yz}$, and $f(t) \in \{x, y, z\}$.

Proof. The argument follows the same pattern as the proof of Lemma 5.2, with (F2) in place of (F1), so it is left to the reader. □

Lemma 5.6. For any distinct $x, y, z \in A$, $Q(x, y) + Q(y, z) \subseteq Q(x, z)$.

Proof. Consider $\alpha \in Q(x, y)$ and $\beta \in Q(y, z)$. Let $t$ be such that $d_t(x, y) \ll \alpha$ and $f(t) = x$, and let $t'$ be such that $d_{t'}(y, z) \ll \beta$ and $f(t') = y$. Lemma 5.5 gives a profile $\hat{t} \in T$ such that $d_{\hat{t}}(x, y) \ll d_t(x, y)$, $d_{\hat{t}}(y, z) \ll d_{t'}(y, z)$, and $f(\hat{t}) \in \{x, y, z\}$. Then negative unanimity implies that $f(\hat{t}) \neq y$ and $f(\hat{t}) \neq z$, so $f(\hat{t}) = x$; since $d_{\hat{t}}(x, z) = d_t(x, y) + d_{\hat{t}}(y, z) \ll \alpha + \beta$, the result is proven. □

Lemma 5.7. $q \in D(A)$.

Proof. Since $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ and $Q(x, z) + Q(z, y) \subseteq Q(x, y)$, it must be the case that $q(x, y) + q(y, z) \leq q(x, z)$ and $q(x, z) + q(z, y) \leq q(x, y)$. But $q(z, y) = -q(y, z)$. Thus, $q(x, y) + q(y, z) = q(x, z)$, as desired. □

For distinct $x, y \in A$, let $Q^*(x, y) = Q(x, y) + q(x, y)e$.

Lemma 5.8. For all distinct $x, y, z$ in $A$ we have:

(a) $Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$.

(b) $Q^*(x, y) \cap -Q^*(y, x) = \emptyset$.

(c) If $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \gg \beta$, then $\alpha \in Q^*(x, y)$ or $\beta \in -Q^*(y, x)$.

(d) $Q^*(x, y) = Q^*(x, z)$.

(e) $Q^*(y, z) = Q^*(x, z)$.

(f) $Q^*(x, y) = Q^*(z, x)$.

(g) $Q^*(x, y) = Q^*(y, x)$.

Proof. (a) This follows from $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ (Lemma 5.6) and $q(x, y) + q(y, z) = q(x, z)$ (Lemma 5.7).

(b) This follows from $Q(x, y) \cap -Q(y, x) = \emptyset$ (Lemma 5.1) and $q(x, y) = -q(y, x)$ (Lemma 5.4).

(c) Since $q(x, y) = -q(y, x)$, Lemma 5.3 implies that $\alpha - q(x, y)e \in Q(x, y)$ or $\beta + q(y, x)e \in -Q(y, x)$.

(d) Since $q(y, z)$ is in the boundary of $Q(y, z)$, the origin is in the boundary of $Q^*(y, z)$, and $Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$, so $Q^*(x, y) \subseteq Q^*(x, z)$. If $\alpha \in Q^*(x, y)$, there is $\beta \in Q^*(x, y)$ with $\beta \ll \alpha$ and $\gamma \in Q^*(x, z)$ arbitrarily close to $\beta$. Therefore there is $\gamma \in Q^*(x, z)$ with $\gamma \ll \alpha$, which implies that $\alpha \in Q^*(x, z)$. 

(e) The argument is the proof of (d) with the roles of $Q^*(x, y)$ and $Q^*(y, z)$ reversed.

Now (d) and (e) give $Q^*(x, y) = Q^*(z, y) = Q^*(z, x) = Q^*(y, x)$, establishing (f) and (g).

Proof of Proposition 5. If $w, x, y, z \in A$ with $x \neq y$ and $w \neq z$, then (d)-(g) imply that $Q^*(x, y) = Q^*(w, z)$ except when the four alternatives are distinct, and in that case (d) and (e) give $Q^*(x, y) = Q^*(x, z) = Q^*(w, z)$. Therefore there is a single set $Q^*$ such that $Q^*(x, y) = Q^*$ for all distinct $x, y$.

We claim that $Q^*$ is convex. It suffices to show that $\frac{1}{2} \alpha + \frac{1}{2} \beta \in Q^*$ whenever $\alpha, \beta \in Q^*$ because for any $0 \leq t \leq 1$ we can find $s$ arbitrarily close to 1 such that

$$(1 - t) \alpha + t \beta = (1 - r) \alpha + r(\alpha + s(\beta - \alpha))$$

where $r$ is an integer multiple of $2^{-k}$ for some positive integer $k$, and $\alpha + s(\beta - \alpha) \in Q^*$ when $s$ is sufficiently close to 1 because $Q^*$ is open. In turn, since $Q^* + Q^* \subset Q^*$ it suffices to show that $\frac{1}{2} \alpha \in Q^*$ whenever $\alpha \in Q^*$. But if $\frac{1}{2} \alpha \notin Q^*$, then for $\beta \ll \frac{1}{2} \alpha$ we have $\beta \in -Q^*$ and $2\beta \in -Q^*$, which implies that $\alpha$ is in the closure of $-Q^*$, and this is impossible because $Q^*$ is open.

Now the separating hyperplane theorem gives a nonzero $\sigma \in \mathbb{R}^n$ such that $\sigma \cdot \alpha > 0$ for all $\alpha \in Q^*$. Since $Q^* + \mathbb{R}^n_{++} \subset Q^*$, all the components of $\sigma$ are nonnegative, so we can take $\sigma \in \Delta^{n-1}$. It must be the case that $\alpha \in Q^*$ whenever $\sigma \cdot \alpha > 0$ because otherwise $\beta \in -Q^*$ whenever $\beta \ll \alpha$, and for $\beta$ close to $\alpha$ we would have $\sigma \cdot \beta > 0$, which is impossible. Thus $Q^* = \{\alpha \in \mathbb{R}^n : \sigma \cdot \alpha > 0\}$.

Suppose that $t \in T$ and $a \in A \setminus \{f(t)\}$. Then $d_t(f(t), a)$ is in the closure of $Q(f(t), a)$, so $d_t(f(t), a) + q(f(t), a)e$ is in the closure of $Q^*$, and consequently

$$0 \leq \sigma \cdot (d_t(f(t), a) + q(f(t), a)e) = \sigma \cdot d_t(f(t), a) + q(f(t), a).$$

Thus $f$ is an affine maximizer.

6. Concluding Remarks

We have corrected and generalized Roberts theorem: if $T$ allows elevation of pairs and is flexible, then the SCF $f$ is truthfully implementable if and only if it is a constant function, binary implementable, or a lexicographic affine maximizer. A flexible domain that allows elevation of pairs may be smaller than the unrestricted quasi-linear domain (even with finite alternative sets). Indeed, we show that the related notion of comprehensiveness is satisfied if the set of alternatives $A$ is a topological space and the domain of admissible valuations is the
space of continuous functions; or if \( A \) is a convex subset of a finite dimensional Euclidean space and the domain of valuations is the space of continuous piecewise affine functions. In the Appendix we show that this is also the case if \( A \) is a compact differentiable manifold and the domain of admissible valuations is the space of smooth functions.

Other characterization of truthful implementability available in the literature are based on cyclic monotonicity a l’a Rochet (1987) or one of its weaker forms. If the set of alternatives is finite, then weak monotonicity characterizes truthful implementation in order-based, auction-like domains (Bikhchandani, Chatterji, Lavi, Mu’alem, Nisan, and Sen (2006b)), convex domains (Saks and Yu (2005)), and monotone domains (Ashlagi, Braverman, Hassidim, and Monderer (2010)). With infinite allocation sets and quasi-linear preferences parameterized by multi-dimensional types, weak monotonicity (in addition to an integral path-independence condition) is sufficient for truthful implementation when valuations are linear in types (Archer and Kleinberg (2008)), while integral weak monotonicity (plus the path-integrability condition) suffices when valuations are convex or differentiable in types (Berger, Müller, and Naeemi (2010)), or Lipschitz continuous in types (Carbajal and Ely (2010)).

These results do not provide a description, in terms of a functional form, of implementable choice functions, which is given by monotonicity in differences in comprehensive domains. It is known that non affine maximizer choice functions can be truthfully implemented in settings with single-dimensional type spaces (cf. Nisan (2007) and references therein). Clearly, single dimensional type domains are not comprehensive. However, the relationship between monotonicity in differences and weak person-by-person monotonicity in multi-dimensional parametric, quasi-linear utility settings remains to be studied.

References


We refer the reader to the original sources for detailed discussions of these concepts and theorems.
Appendix: Smooth Valuations

At the cost of somewhat greater complications, it is possible to show that spaces of smooth valuations are flexible and allow elevation of pairs. To do so we replace (C2) with the following weaker condition:

\[(C2^*)\] For all \(i \in N\), \(v_i, v'_i \in V_i\), and \(x, y \in A\) such that \(v_i(x) \neq v'_i(x)\) and \(v_i(y) \neq v'_i(y)\) there is a \(\hat{v}_i \in V_i\) such that

\[
\hat{v}_i(a) \leq v_i(v'_i(a)), \quad \text{for all } a \in A,
\]

with equality when \(a = x\) and when \(a = y\).

Given a separating family \(U\) of functions from \(A\) to \([0, 1]\), the domain \(V\) is said to be \(U\)-comprehensive* if (C1), (C2*) and (C3) are satisfied. Two tasks require attention. First, we explain the sense in which spaces of smooth functions on a compact manifolds are comprehensive*. Second, we show that the arguments in Section 3 can be amended to take into account the weakening of (C2) to (C2*).

Fix an order of differentiability \(1 \leq r \leq \infty\). We now assume that \(A\) is a compact \(C^r\) manifold, and as usual we let \(C^r(A)\) be the space of all \(C^r\) real valued functions on \(A\). We will often describe elements of \(C^r(A)\) as smooth functions.

**Proposition A.1.** If, for every agent \(i \in N\), \(V_i = C^r(A)\), then \(V\) is \(C^r(A)\)-comprehensive*.

**Proof.** (C1) Consider \(i \in N\), \(v_i \in V_i\), \(\mu \in C^r(A)\) mapping \(A\) to \([0, 1]\), \(x \in A\), and \(\delta_i \in \mathbb{R}\). Standard constructions give an element of \(C^r(A)\) whose support is compact, and whose maximum is attained uniquely at \(x\). (The existence of such smooth functions, and others chosen below, follows from standard constructions, cf. Section 2.2 of Hirsch (1976).) Subtracting the constant function whose value is the value of this function at \(x\) gives \(\nu \in C^r(A)\) with \(\nu(x) = 0\) and \(\nu(a) < 0\) for all \(a \neq x\). It
then suffices to set

$$\hat{v}_i = v_i + \delta_i \mu + \nu.$$  

(C2*) Let $v, v' \in V$ and $x, y \in A$ be such that for each $i \in N$, $v_i(x) \neq v'_i(x)$ and $v_i(y) \neq v'_i(y)$. Let $B_x$ and $B_y$ be disjoint neighborhoods of $x$ and $y$, respectively, such that for each agent $i$, the sign of $v_i(a) - v'_i(a)$ coincides with the sign of $v_i(x) - v'_i(x)$ for all $a \in B_x$, and similarly the sign of $v_i(a) - v'_i(a)$ coincides with that of $v_i(y) - v'_i(y)$ for all $a \in B_y$. Let $c = (c_1, \ldots, c_n) \in V$ be a profile of constant functions such that $c_i \leq v_i \wedge v'_i$, for all $i = 1, \ldots, n$, which exists since $A$ is compact. We can also find a smooth function $\nu: A \to [0, 1]$ such that $\nu(x) = \nu(y) = 1$, $\nu(a) < 1$ if $a \neq x, y$, and $\nu(a) = 0$ if $a \notin B_x \cup B_y$. Define the function profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ by

$$\hat{v}_i(a) = (1 - \nu(a))c_i + \nu(a) \min\{v_i(a), v'_i(a)\}.$$  

The valuation profile $\hat{v} \in V$ satisfies the required properties.

(C3) Fix $v, v', v'' \in V$ and three distinct alternatives $x, y, z \in A$. Let $B_x, B_y, B_z$ be mutually disjoint neighborhoods of $x, y, z$, respectively. As in the preceding argument, let $c = (c_1, \ldots, c_n)$ be a profile of constant functions satisfying $c_i \leq v_i \wedge v'_i \wedge v''_i$ for all $i \in N$, and let $\nu: A \to [0, 1]$ be a smooth function such that $\nu(a) = 1$ if $a = x, y, z$, $\nu(a) < 1$ otherwise, and $\nu(a) = 0$ for all $a \notin (B_x \cup B_y \cup B_z)$. Consider now the profile of valuations $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ such that for each $i$, $\hat{v}_i$ is defined on $A$ by

$$\hat{v}_i(a) = \begin{cases} (1 - \nu(a))c_i + \nu(a)v_i(a), & \text{if } a \in B_x, \\ (1 - \nu(a))c_i + \nu(a)v'_i(a), & \text{if } a \in B_y, \\ (1 - \nu(a))c_i + \nu(a)v''_i(a), & \text{if } a \in B_z, \\ c_i, & \text{otherwise.} \end{cases}$$

Clearly, $\hat{v}$ is smooth and satisfies the desired conditions.  

We show next that the comprehensiveness* of $V = C^r(A)^n$ implies that $V$ is flexible and allows elevation of pairs.

**Lemma A.2.** If $V = C^r(A)^n$ is $C^r(A)$-comprehensive*, then for any set $K \subset A$ with either two or three elements and any valuation profile $v^x \in V$ indexed for the various $x \in K$, there exists a profile of admissible valuations $v \in V$ such that $v \leq \bigwedge_{x \in K} v^x$.

**Proof.** Immediate from the fact that $V_i = C^r(A)$, for all $i \in N$.  

**Lemma A.3.** Suppose that $V = C^r(A)^n$ is $C^r(A)$-comprehensive* and let $K \subset A$ contain two or three alternatives. Then for any valuation profiles $v^x$ for the various $x \in K$, there are pairwise disjoint sets $B_x \subset
Proof. Fix valuations \( v^x \in V \) for the various \( x \in K \). Let \( \{ \mu^x \mid x \in K \} \) be a collection of smooth separating functions from \( A \) to \([0,1] \), so that \( \mu^x(x) = 1 \) for each \( x \in K \) and the supports \( B_x = \text{supp}(\mu^x) \) are pairwise disjoint. Choose \( \tilde{v} \in V \) such that \( \tilde{v} \leq \bigwedge_{x \in K} v^x \).

As in the proof of Lemma 3.4, for each \( i \in N \) choose \( \delta_i > 0 \) large enough that \( \tilde{v}_i(x) + \delta_i > v^x_i(x) \) for every \( x \in K \). For given \( x \in K \), (C1) implies that there is a \( \tilde{v}^x_i \in V_i \) such that \( \tilde{v}^x_i(x) = \tilde{v}_i(x) + \delta_i \) and \( \tilde{v}^x_i(a) < \tilde{v}_i(a) + \delta_i \mu^x(a) \) for all \( a \in A \), \( a \neq x \). Since \( \tilde{v}^x_i(x) > v^x_i(x) \), apply (C2*) for \( y = x \) to conclude that there is a \( \tilde{v}^x_i \in V_i \) such that \( \tilde{v}^x_i(x) = v^x_i(x) \) and \( \tilde{v}^x_i(a) \leq \tilde{v}^x_i(x) \) for all \( a \in A \). The rest of the argument follows the same lines of the proof of Lemma 3.4.

We have the following corollary.

**Corollary A.4.** If the domain of valuation profiles \( V = C^r(A)^n \) is \( C^r(A) \)-comprehensive*, then it is flexible.

**Proof.** Apply the argument of Corollary 3.5 with Lemma A.3 replacing Lemma 3.4.

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**Lemma A.5.** Suppose that the domain of valuation profiles \( V = C^r(A)^n \) is \( C^r(A) \)-comprehensive*. If \( v, v' \in V \) and \( x, y \in A \) satisfy \( d_v(x,y) \gg d_v(x,y) \), then there exists \( \hat{v} \in V \) such that \( d_v(x,a) \gg d_v(x,a) \) for all \( a \neq x \) and \( d_v(y,a) \gg d_v(y,a) \) for all \( a \neq y \).

**Proof.** Since \( d_v(x,y) \gg d_v(x,y) \), one has \( v'_i(x) - v_i(x) > v'_i(y) - v_i(y) \) for all \( i \in N \). If for \( i \in N \) it is that \( v'_i(x) - v_i(x) > v'_i(y) - v_i(y) \), then from (C2*) there exists a valuation \( \tilde{v} \in V_i \) such that \( \tilde{v}_i(x) = v'_i(x), \tilde{v}_i(y) = v'_i(y) \), and \( \tilde{v}_i(a) \leq v_i(a) \) for all \( a \neq x, y \).

If instead \( 0 \geq v'_i(x) - v_i(x) > v'_i(y) - v_i(y) \), choose \( \delta_i > 0 \) so that \( v'_i(x) + \delta_i - v_i(x) > 0 > v'_i(y) + \delta_i - v_i(y) \). Letting \( v''_i = v_i + \delta_i \in V_i \), we apply (C2*) to obtain a valuation \( \tilde{v} \in V_i \) such that \( \tilde{v}_i(x) = v''_i(x), \tilde{v}_i(y) = v''_i(y) \), and \( \tilde{v}_i(a) \leq v_i \vee v''_i(a) \) for all \( a \in A, a \neq x, y \). Notice that \( d_v'' = d_v' \). The case where \( v'_i(x) - v_i(x) > v'_i(y) - v_i(y) \geq 0 \) is similarly handled.

The rest of the argument follows the lines of the proof of Lemma 3.6. We omit details. 

\[ \square \]
Corollary A.6. If $V = \mathcal{C}^r(A)^n$ is $\mathcal{C}^r(A)$-comprehensive*, then it allows elevation of pairs.

Proof. Since $V$ is a cartesian product of $\mathcal{C}^r(A)$, this follows immediately from Lemma A.5. \qed

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