Games with Discontinuous Payoffs: a Strengthening of Reny’s Existence Theorem

Andrew McLennan, Paulo K. Monteiro, and Rabee Tourky

December 1, 2010

Abstract: We provide a pure Nash equilibrium existence theorem for games with discontinuous payoffs whose hypotheses are in a number of ways weaker than those of the theorem of Reny (1999). In comparison with Reny’s argument, our proof is brief. Our result subsumes a prior existence result of Nishimura and Friedman (1981) that is not covered by his theorem. We use the main result to prove existence of pure Nash equilibrium in a class of finite games in which agents’ pure strategies are subsets of a given set, and in turn use this to prove the existence of stable configurations for games, similar to those used by Schelling (1971, 1972) to study residential segregation, in which agents choose locations.

1 Introduction

Many important and famous games in economics (e.g, the Hotelling location game, Bertrand competition, Cournot competition with fixed costs, and various auction models) have discontinuous payoffs, and consequently do not satisfy the hypotheses of Nash’s existence proof or its infinite dimensional generalizations, but nonetheless have at least one pure Nash equilibria. Using an argument that is quite ingenious and involved, Reny (1999) establishes a result that subsumes earlier equilibrium existence results covering many such examples. His theorem’s hypotheses are simple and weak, and in many cases easy to verify. The result has been applied in

*We have benefitted from useful conversations with Paulo Barelli, Marcus Berliant, Philippe Bich, Priscilla Man, Claudio Mezzetti, Stuart McDonald, Rohan Pitchford, Pavlov Prokopovych, and Phil Reny. We are also grateful for useful comments and suggestions from Luciano de Castro, Nikolai Kukushkin, the editor, and anonymous referees. Comments of seminar audiences at the University of Chicago, the University of Illinois Urbana-Champaign, the University of Kyoto, the University of Minnesota, Université de Paris I, the 2009 European Workshop on General Equilibrium Theory, the 2009 NSF/NBER/CEME Conference on General Equilibrium and Mathematical Economics, the Tenth SAET Conference, and the 2nd UECE - Lisbon Meetings, are gratefully acknowledged. McLennan’s work was funded in part by Australian Research Council grant DP0773324 and Tourky’s work was in part funded by Australian Research Council grant DP1093105.
novel settings many times since then. (See for example Monteiro and Page (2008).) Largely in response to his work, a number of papers on discontinuous games have appeared recently (Carmona (2005); Bagh and Jofre (2006); Bich (2006, 2009); Monteiro and Page (2007); Carmona (2009); Carbonell-Nicolau (2011, 2010a,b); Prokopovych (2010a,b); Barelli and Soza (2009); de Castro (2011); Tian (2009)). Of these, Barelli and Soza (2009) deserves special mention because it adopts many techniques from an earlier version of this paper. Briefly, a key idea in Reny (1999) and here is “securing a payoff” at a strategy profile for the other players by playing a pure strategy that insures that payoff when the profile of other players’ strategies is near the given profile. Barelli and Soza show that our main result continues to hold if one allows the securing strategy to be a continuous function of the profile of other players’ strategies.

As Reny explains, the main result of Nishimura and Friedman (1981) and results concerning the existence of Cournot equilibria (Szikaróvzy and Yakowitz (1977) and the independent (see Roberts and Sonnenschein (1977)) work of McManus (1962, 1964) and Roberts and Sonnenschein (1976)) seemingly have a different character, and are not obvious consequences of his result. Here we provide a generalization of Reny’s theorem that easily implies the Nishimura and Friedman result.

The refinements of Reny’s theorem introduced in Section 3 are of two sorts: (a) we weaken the condition of better reply security at a point by allowing several securing strategies to be used; (b) we allow subcorrespondences of the better reply correspondence to be specified by restricting the agents to subsets of their sets of pure strategies. In Section 5 we present an application of (a) and (b). We prove existence of pure Nash equilibria for a class of finite games in which each agent chooses from a collection of subsets of a given finite set, and of course payoffs are restricted in a certain way. Section 6 uses this result to establish the existence of pure Nash equilibria for a model in which agents choose locations that is similar in spirit to the games that Schelling (1971, 1972) uses to study residential segregation. Whereas Schelling uses simulation to study the qualitative properties of myopic (in the sense that future relocations are unanticipated) adjustment dynamics, our result demonstrates the existence of stable configurations, which may be viewed as the possible rest points for the sorts of processes he studies.

The remainder has the following organization. The next section reviews Reny’s theorem and states a result whose hypotheses are less restrictive than Reny’s, but more restrictive than our main result. We also explain how this result implies the Nishimura-Friedman existence theorem. Section 3 states our main result. After a required combinatoric concept has been introduced in Section 4, Section 5 presents the results concerning games on subsets. Section 6 present the location model and uses a particular instance of it to demonstrate that other methods of proving existence of pure Nash equilibria are inapplicable. Section 7 presents the proof of Theorem 3.4. Reny introduces two concepts, payoff security and reciprocal upper
semicontinuity, which together imply the hypotheses of his main result, and which are often easy to verify in applications. Section 8 explains the extension of those concepts to our setting. Section 9 contains some concluding remarks and thoughts about possible extensions.

2 Reny and Nishimura-Friedman

Our system of notation is largely taken from Reny (1999). There is a fixed normal form game

$$G = (X_1, \ldots, X_N, u_1, \ldots, u_N)$$

where, for each \(i = 1, \ldots, N\), the \(i\)-th player’s strategy set \(X_i\) is a nonempty compact convex subset of a topological vector space, and the \(i\)-th player’s payoff function \(u_i\) is a function from the set of strategy profiles \(X = \prod_{i=1}^{N} X_i\) to \(\mathbb{R}\).

We adopt the usual notation for “all players other than \(i\).” Let \(X_{-i} = \prod_{j \neq i} X_j\). If \(x \in X\) is given, \(x_{-i}\) denotes the projection of \(x\) on \(X_{-i}\). For given \(x_{-i} \in X_{-i}\) (or \(x \in X\)) and \(y_i \in X_i\) we write \((y_i, x_{-i})\) for the strategy profile \(z \in X\) satisfying \(z_i = y_i\) and \(z_j = x_j\) for all \(j \neq i\). We endow \(X\) and each \(X_{-i}\) with their product topologies. These conventions will also apply to other strategic form games introduced later.

We now review Reny’s theorem. A Nash equilibrium of \(G\) is a point \(x^* \in X\) satisfying \(u_i(x^*) \geq u_i(y_i, x^*_{-i})\) for all \(i\) and all \(y_i \in X_i\). (This would usually be described as a “pure Nash equilibrium,” but we never refer to mixed equilibria, so we omit the qualifier ‘pure.’) For each player \(i\) let \(B_i : X \times \mathbb{R} \to X_i\) and \(C_i : X \times \mathbb{R} \to X_i\) be the set valued mappings

$$B_i(x, \alpha_i) = \{y_i \in X_i : u_i(y_i, x_{-i}) \geq \alpha_i\} \quad \text{and} \quad C_i(x, \alpha_i) = \text{con} B_i(x, \alpha_i).$$

(Here and below con \(S\) is the convex hull of the set \(S\).) Then \(u_i\) is quasiconcave if and only if \(C_i(x, \alpha_i) = B_i(x, \alpha_i)\) for all \(x \in X\) and \(\alpha_i \in \mathbb{R}\). We say that \(G\) is quasiconcave if each \(u_i\) is quasiconcave.

**Definition 2.1.** A player \(i\) can secure a payoff \(\alpha_i \in \mathbb{R}\) on \(S \subseteq X\) if there is some \(y_i \in X_i\) such that \(y_i \in B_i(z, \alpha_i)\) for all \(z \in S\). We say that \(i\) can secure \(\alpha_i\) at \(x \in X\) if she can secure \(\alpha_i\) on some neighborhood of \(x\).

Throughout we assume that each \(u_i\) is bounded. Let \(u = (u_1, \ldots, u_N) : X \to \mathbb{R}^N\). For \(x \in X\) let \(A(x)\) be the set of \(\alpha \in \mathbb{R}^N\) such that \((x, \alpha)\) is in the closure of the graph of \(u\). Since \(u\) is bounded, each \(A(x)\) is compact.

**Definition 2.2.** The game \(G\) is better reply secure at \(x \in X\) if, for any \(\alpha \in A(x)\), there is some player \(i\) and \(\varepsilon > 0\) such that player \(i\) can secure \(\alpha_i + \varepsilon\) at \(x\). The game \(G\) is better reply secure if it is better reply secure at every strategy profile that is not a Nash equilibrium.
Theorem 2.3 (Reny (1999)). If \( G \) is quasiconcave and better reply secure, then it has a Nash equilibrium.

To better understand Reny’s result in the context of this paper we reformulate better reply security.

**Definition 2.4.** The game is **B-secure on** \( S \subset X \) if there is \( \alpha \in \mathbb{R}^N \) and \( \varepsilon > 0 \) such that:

(a) every player \( i \) can secure \( \alpha_i + \varepsilon \) on \( S \);

(b) for each \( z \in S \) there exists some player \( i \) with \( u_i(z) < \alpha_i - \varepsilon \), i.e., \( z \notin B_i(z, \alpha_i - \varepsilon) \).

The game is **B-secure at** \( x \in X \) if it is B-secure on some neighborhood of \( x \).

**Lemma 2.5.** For each \( x \in X \) the game is better reply secure at \( x \) if and only if it is B-secure at \( x \).

**Proof.** First assume that the game is B-secure at \( x \), with \( \alpha \), \( \varepsilon \) and \( U \) as in the definition. Each \( \alpha' \in A(x) \) is the limit of values of \( u \) along some sequence or net converging to \( x \), so there is some \( i \) with \( \alpha'_i \leq \alpha_i - \varepsilon \). This \( i \) can secure \( \alpha'_i + \varepsilon \) at \( x \), which shows that the game is better reply secure at \( x \).

Now assume that the game is better reply secure at \( x \). Let \( \tau \) be a neighborhood base of \( x_{-i} \), and let

\[
\beta_i = \sup_{y_i \in X_i} \sup_{U \in \tau} \inf_{z \in U} u_i(y_i, z_{-i}).
\]

Then \( \alpha'_i < \beta_i \) if and only if there is some \( \varepsilon > 0 \) such that \( \alpha'_i + \varepsilon \) can be secured by player \( i \) at \( x \). Since the game is better reply secure at \( x \), for each \( \alpha' \in A(x) \) there is some player \( i \) such that \( \beta_i > \alpha'_i \), which implies that the inequality \( \beta_i > \alpha''_i + \varepsilon \) holds for some \( \varepsilon > 0 \) and all \( \alpha'' \) in some neighborhood of \( \alpha' \). Since \( A(x) \) is compact, it is covered by finitely many such neighborhoods, so we may choose \( \varepsilon > 0 \) such that for any \( \alpha' \in A(x) \) there is some \( i \) with \( \beta_i > \alpha'_i + 2\varepsilon \). Define \( \alpha \in \mathbb{R}^N \) by setting

\[
\alpha_i = \beta_i - \varepsilon.
\]

In view of the definition of \( \beta_i \), for each \( i \), player \( i \) can secure \( \alpha_i + \varepsilon \) at \( x \), as per (a).

Aiming at a contradiction, suppose (b) is false, so for each \( U \in \tau \) there is some \( z_U \in U \) such that \( u_i(z_U) \geq \alpha_i - \varepsilon \) for all \( i \). Since \( \tau \) is a directed set (ordered by reverse inclusion) the boundedness of the image of \( u \) implies that there is a convergent subnet, so there is \( \alpha' \in A(x) \) such that \( \alpha'_i \geq \alpha_i - \varepsilon = \beta_i - 2\varepsilon \) for all \( i \), contrary to what we showed above.

When \( G \) is quasiconcave the following condition is weaker than B-security.
**Definition 2.6.** The game is $C$-secure on $S \subset X$ at $x \in X$ if there is an $\alpha \in \mathbb{R}^N$ such that:

(a) every player $i$ can secure $\alpha_i$ on $S$;

(b) for any $z \in S$ there exists some player $i$ with $z_i \notin C_i(z, \alpha_i)$.

The game is $C$-secure at $x \in X$ if it is $C$-secure on some neighborhood of $x$.

A simple maximization example illustrates how getting rid of the $\varepsilon$ may matter. Let $N = 1$ and $X_1 = [0, 1]$ with $u_1(x_1) = \begin{cases} 0, & x_1 = 0, \\ (x_1 - \frac{1}{2})^2, & 0 < x_1 \leq 1. \end{cases}$ Setting $\alpha_1 = 1/4$, we see that the game is $C$-secure at every $x_1 \in [0, 1)$, but it can easily be checked that the game is not $B$-secure at the point $x_1 = 0$, which is not a Nash equilibrium, so it is not better reply secure.

In comparison with Definition 2.4, part (b) Definition 2.6 replaces $B_i(z, \alpha_i)$ with $C_i(z, \alpha_i)$, which makes it harder to satisfy in general, but not when $G$ is quasiconcave, so the net effect is to make the hypotheses of the next result weaker than those of Theorem 2.3. The hypotheses of Theorem 3.4 will be weaker still.

**Proposition 2.7.** If the game is $C$-secure at each $x \in X$ that is not a Nash equilibrium, then $G$ has a Nash equilibrium.

Nishimura and Friedman (1981) prove the existence of a Nash equilibrium when each $X_i$ is a nonempty, compact, convex subset of a Euclidean space, $u$ is continuous (but not necessarily quasiconcave) and for any $x$ that is not a Nash equilibrium there is an agent $i$, a coordinate index $k$, and an open neighborhood $U$ of $x$, such that

$$(y^1_{ik} - x^1_{ik})(y^2_{ik} - x^2_{ik}) > 0$$

whenever $x^1, x^2 \in U$ and $y^1_i$ and $y^2_i$ are best responses for $i$ to $x^1$ and $x^2$ respectively. Using compactness, and the continuity of $u_i$, it is not difficult to show that this is equivalent to $(y^1_{ik} - x_{ik})(y^2_{ik} - x_{ik}) > 0$ for any two best responses $y^1_i, y^2_i$ to $x$. A more general condition that does not depend on the coordinate system, or the assumption of finite dimensionality, is that there is a hyperplane that strictly separates $x_i$ from the set of $i$’s best responses to $x$.

We now show that if $G$ satisfies the hypotheses of Nishimura and Friedman’s result, then it is $C$-secure and thus satisfies the hypotheses of Proposition 2.7. Consider an $x \in X$ that is not a Nash equilibrium. For each $i = 1, \ldots, N$ let $\beta_i$ be the utility for $i$ when other agents play their components of $x_{-i}$ and $i$ plays a best response to $x$. Since $u$ is continuous, for any $\varepsilon > 0$ player $i$ can secure $\beta_i - \varepsilon$ at $x$ by playing such a best response. For any neighborhood $V$ of $B_i(x, \beta_i)$ it is the case
that \(B_i(x, \beta_i - \epsilon) \subset V\) when \(\epsilon\) is sufficiently small, and in turn it follows that there is a neighborhood \(U\) of \(x\) such that \(B_i(z, \beta_i - \epsilon) \subset V\) for all \(z \in U\). It follows that if there is a hyperplane strictly separating \(x_i\) from \(B_i(x, \beta_i)\), then for sufficiently small \(\epsilon > 0\) and a sufficiently small neighborhood \(U\) of \(x\), this hyperplane also strictly separates \(z_i\) and \(B_i(z, \beta_i - \epsilon)\) for all \(z \in U\), in which case \(z_i \notin C_i(z, \beta_i)\). Setting \(\alpha = (\beta_1 - \epsilon, \ldots, \beta_N - \epsilon)\) gives the required property.

3 The Main Result

Our main result weakens the hypotheses of Proposition 2.7 in two ways. The first is to allow multiple securing strategies at a point, with each strategy responsible for securing the payoff in response to profiles in some component of a finite closed cover of a neighborhood of the point. The second weakening is that the analyst is allowed to specify restrictions on the strategies that an agent is allowed to consider in response to a profile. We introduce these in the next definition.

For each player \(i\) fix a set valued mapping \(X_i: X \to X_i\), and let \(X = (X_1, \ldots, X_N)\). We call \(X\) a restriction operator. For each \(i\) define the set valued mappings \(B_i^X: X \times \mathbb{R} \to X_i\) and \(C_i^X: X \times \mathbb{R} \to X_i\) by setting

\[
B_i^X(x, \alpha_i) = \{y_i \in X_i(x): u_i(y_i, x_{-i}) \geq \alpha_i\} \quad \text{and} \quad C_i^X(x, \alpha_i) = \text{con} B_i^X(x, \alpha_i).
\]

**Definition 3.1.** Player \(i\) can multiply \(X\)-secure a payoff \(\alpha_i \in \mathbb{R}\) on a set \(S \subset X\) if \(S\) has a finite cover \(F^1, \ldots, F^J\) by relatively closed sets such that for each \(j\) there is some \(y^j_i \in X_i\) satisfying \(y^j_i \in B_i^X(z, \alpha_i)\) for all \(z \in F^j\). Player \(i\) can multiply \(X\)-secure \(\alpha_i\) at \(x \in X\) if she can multiply \(X\)-secure \(\alpha_i\) on a neighborhood of \(x\).

**Definition 3.2.** If \(\alpha \in \mathbb{R}^N\) and \(S \subset X\), the game \(G\) is multiply \((\mathcal{X}, \alpha)\)-secure on \(S \subset X\) if:

(a) each player \(i\) can multiply \(X\)-secure \(\alpha_i\) on \(S\);

(b) for any \(z \in S\) there is some player \(i\) for whom \(z_i \notin C_i^X(z, \alpha_i)\).

The game is multiply \(\mathcal{X}\)-secure on \(S\) if there is some \(\alpha \in \mathbb{R}^n\) such that it is multiply \((\mathcal{X}, \alpha)\)-secure on \(S\), and it is multiply \(\mathcal{X}\)-secure at \(x\) if it is multiply \(\mathcal{X}\)-secure on some neighborhood of \(x\).

Following Reny (1999) we do not require that the \(X_i\) are Hausdorff spaces. (As Reny acknowledges, this is a mathematical refinement without any known economic applications.) Thus, for \(x \in X\) the set \(\{x\}\) need not be closed, and we let \([x]\) denote the closure of \(\{x\}\).

**Definition 3.3.** The game \(G\) is multiply \(\mathcal{X}\)-secure if the set of \(x\) such that \([x]\) does not contain a Nash equilibrium is covered by open sets \(U\) such that \(G\) is multiply \(\mathcal{X}\)-secure on \(U\). We say that \(G\) is multiply restrictionally secure, or MR secure, if there is a restriction operator \(\mathcal{X}\) such that it is multiply \(\mathcal{X}\)-secure.
We say that \( X \) is the universal restriction operator if \( X_i(x) = X_i \) for all \( i \) and \( x \in X \). If \( G \) is multiply \( X \)-secure at \( x \) (multiply \( X \)-secure) for this particular restriction operator, then we say simply that it is multiply secure at \( x \) (multiply secure). Similarly, we drop the qualifier ‘multiply’ if all the relevant closed covers have a single element.

Our main result is as follows:

**Theorem 3.4.** If \( G \) is MR secure, then it has a Nash equilibrium.\(^1\)

The reader may proceed directly to the proof, which is in Section 7.

### 4 Realizable Pairs

In preparation for the games considered in the next section we introduce a concept related to combinatoric geometry. Let \( S \) be a nonempty finite set, let \( A \) be a nonempty set of nonempty subsets of \( S \), and let \( B \) be a set of subsets of \( S \).

**Definition 4.1.** The pair \((A, B)\) is realizable if there exists a compact convex subset \( X \) of a finite dimensional vector space, a continuous function \( f : X \to \mathbb{R}^S \), and convex sets \( Y_b \subset X \) for the \( b \in B \), such that

(a) \( A = \{ \arg\max_{s \in S} f_s(x) : x \in X \} \);

(b) \( B \) is closed under intersection;

(c) if \( b \subset b' \in B \), then \( Y_b \subset Y_{b'} \);

(d) \( \{ a \in A : a \subset b \} = \{ \arg\max_{s \in S} f_s(x) : x \in Y_b \} \) for all \( b \in B \).

We illustrate this concept with a rich class of realizable pairs. Fix an integer \( d \geq 1 \). A simplex \( \sigma \) in \( \mathbb{R}^d \) is the convex hull of an affinely independent set of points \( V_{\sigma} \subset \mathbb{R}^d \), the elements of which are called the vertices of the simplex. A face of \( \sigma \) is the convex hull of a (possibly empty or nonproper) subset of \( V_{\sigma} \). A (finite) simplicial complex is a finite collection of simplices \( \Sigma \) such that:

(a) \( \tau \in \Sigma \) whenever \( \sigma \in \Sigma \) and \( \tau \) is a face of \( \Sigma \);

(b) for any \( \sigma, \sigma' \in \Sigma \), \( \sigma \cap \sigma' \) is a face of both \( \sigma \) and \( \sigma' \).

Fix such a \( \Sigma \), and let \( X = \bigcup_{\sigma \in \Sigma} \sigma \) and \( V = \bigcup_{\sigma \in \Sigma} V_{\sigma} \).

Let \( T \) be a second simplicial complex in \( \mathbb{R}^d \). We say that \( \Sigma \) is a subdivision of \( T \) if \( \bigcup_{\tau \in T} \tau = X \) and each \( \sigma \in \Sigma \) is contained in some \( \tau \in T \). Assuming this is the case, for each \( \tau \in T \) let \( W_\tau = \bigcup_{\sigma \subseteq \tau} V_{\sigma} = \tau \cap V \), and let

\[
A = \{ V_{\sigma} : \emptyset \neq \sigma \in \Sigma \} \quad \text{and} \quad B = \{ W_\tau : \tau \in T \}.
\]

---

\(^1\)The converse holds as well, but for a trivial reason: if \( x^* \) is a Nash equilibrium, and \( X_i(x) = \{ x_i^* \} \) for all \( i \) and \( x \), then \( G \) is \( X \)-secure.
Proposition 4.2. If $X$ is convex, then the pair $(\mathcal{A}, \mathcal{B})$ is realizable.

Proof. For $x \in X$ let $f(x)$ be the point in $\mathbb{R}^V$ satisfying $\sum f_w(x) = 1$, $\sum f_w(x)w = x$, and $f_w(x) = 0$ if $w \notin V_{\sigma_x}$, where $\sigma_x$ be the smallest simplex in $\Sigma$ containing $x$. Note that $f : X \to \mathbb{R}^V$ is continuous. For each $x \in X$ we have argmax$_{v \in V} f_v(x) = V_{\sigma_x}$ for some nonempty face $\sigma$ of $\sigma_x$, and for each nonempty $\sigma \in \Sigma$ there is a point $x_{\sigma} \in \sigma$ such that $\arg\max W_{\sigma_{\sigma_x}} f_w(x_{\sigma}) = V_{\sigma}$. Therefore (a) holds.

For $\tau, \tau' \in T$ we have $W_{\tau} \cap W_{\tau'} = \tau \cap \tau' \cap V = W_{\tau \cap \tau'}$, so (b) holds.

That (c) holds is obvious.

For $b = W_{\tau} \in \mathcal{B}$ let $Y_b = \tau$. Consider $a = V_{\sigma} \in \mathcal{A}$ and $b = W_{\tau} \in \mathcal{B}$. If $a \subset b$, then $x_{\sigma} \in \sigma \subset \tau = Y_b$. On the other hand, if $x \in Y_b$, then $\arg\max_{s \in S} f_s(x)$ is the set of vertices of a face of $\sigma_x$, so it is a subset of $b$. Thus (d) holds. ■

For our work the most important example of realizability is:

Corollary 4.3. If $\mathcal{A}$ is the set of all nonempty subsets of a nonempty set $S$ and $\mathcal{B} = \mathcal{A} \cup \{\emptyset\}$, then $(\mathcal{A}, \mathcal{B})$ is realizable.

Proof. Let $V$ be the set of standard unit basis vectors $(0, \ldots, 1, \ldots, 0) \in \mathbb{R}^S$, let $\Sigma = T$ be the set of convex hulls of subsets of $V$, and apply the last result. ■

5 Games on Subsets

We now study a finite game

$$
\Gamma = (\mathcal{A}_1, \ldots, \mathcal{A}_N, v_1, \ldots, v_N)
$$

in which for each $i = 1, \ldots, N$ the set $\mathcal{A}_i$ of pure strategies for agent $i$ is a nonempty set of nonempty subsets of a finite set $S_i$. Set $\mathcal{A} = \prod_{i=1}^N \mathcal{A}_i$. Each $v_i$ is a function from $\mathcal{A}$ to $\mathbb{R}$.

For each $i$ let $\mathcal{B}_i$ be a set of subsets of $S_i$, and let $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$. A $\mathcal{B}$-diagnostic for $\Gamma$ is an $N$-tuple $d = (d_1, \ldots, d_N)$ where each $d_i$ is a function from $\mathcal{A}_{-i}$ to the set of all subsets of $S_i$, such that:

(D1) If $a \in \mathcal{A}$ is not a Nash equilibrium, then there is an $i$ such that $d_i(a_{-i}) \in \mathcal{B}_i$ and $a_i$ is not a subset of $d_i(a_{-i})$.

(D2) If $a^1, \ldots, a^K$ is a sequence in $\mathcal{A}$ that is decreasing, in the sense that $a_i^{k+1} \subset a_i^k$ for all $i$ and $k < K$, or increasing, then for each $i$ there is some $a_i \in \mathcal{A}_i$ such that $a_i \subset \bigcap_{k=1}^K d_i(a_{-i}^k)$.

We say that $\Gamma$ is a game on subsets if, for some $\mathcal{B}$, it has a $\mathcal{B}$-diagnostic.

Theorem 5.1. If $\Gamma$ is a game on subsets and each $(\mathcal{A}_i, \mathcal{B}_i)$ is realizable, then $\Gamma$ has a Nash equilibrium.
Proof. For each $i$ let $E_i = \mathbb{R}^{S_i}$, and let $X_i, Y_{b_i} \subset X_i$ for $b_i \in B_i$, and $f_i : X_i \to E_i$ be the sets and functions from the realizability assumption. Set $X = \prod_{i=1}^{N} X_i$ and $E = \prod_{i=1}^{N} E_i$, and define $f : X \to E$ by setting $f(x) = (f_1(x_1), \ldots, f_N(x_N))$.

For each $i$ let $a_i : X_i \to A_i$ be the function $a_i(x_i) = \arg\max_{s_i \in S_i} f_i(s_i(x_i))$, and define $a : X \to A$ by setting $a(x) = (a_1(x_1), \ldots, a_N(x_N))$. We define a game $G = (X_1, \ldots, X_N, u_1, \ldots, u_N)$ by setting $u_i(x) = v_i(a(x))$. For each $i$ the image of $a_i$ is $A_i$ (by (a) of the definition of realizability) so $x^* \in X$ is a Nash equilibrium of $G$ if and only if $a(x^*)$ is a Nash equilibrium of $\Gamma$.

For each $x \in X$ and player $i$ let

$$\mathcal{X}_i(x) = \begin{cases} Y_{d_i(a_{-i}(x_{-i}))}, & \text{if } d_i(a_{-i}(x_{-i})) \in B_i, \\ X_i, & \text{otherwise.} \end{cases}$$

We will show that if $G$ has no Nash equilibria, it is necessarily multiply $\mathcal{X}$-secure, which is impossible because Theorem 3.4 would imply that it had a Nash equilibrium after all. For each $i$ let $a_i$ be a lower bound on $v_i$, and let $\alpha = (\alpha_1, \ldots, \alpha_N)$. Fix $x \in X$. Our goal in the remainder of the proof is to show that if $G$ has no Nash equilibria, then $G$ is multiply $(\mathcal{X}, \alpha)$-secure at $x$.

For any $z \in X$ (D1) tell us that there exists some $i$ such that $d_i(a_{-i}(z_{-i})) \in B_i$ and $a_i(z_i)$ is not a subset of $d_i(a_{-i}(z_{-i}))$. This implies, by (d) of Definition 4.1, that $z_i \notin \mathcal{X}_i(z) = B_i^X(z, \alpha_i)$. Clearly $\mathcal{X}_i$ is convex valued, so $z_i \notin C_i^X(z, \alpha_i)$. It remains to show that each $i$ can multiply $\mathcal{X}$-secure $\alpha_i$ at $x$.

Let $S$ be the disjoint union of the sets $S_1, \ldots, S_N$. Consider a strict complete ordering $\succ$ of $S$. For each $i$ let $s^*_i$ be the maximal element of $S_i$. We say that $\beta \in E$ conforms to $\succ$ if $s^*_i \in \arg\max_{s_i \in S_i} \beta_{is_i}$ for all $i$ and

$$\beta_{is^*_i} - \beta_{js^*_j} \leq \beta_{js_j} - \beta_{js_j}$$

for all $s_i \in S_i \setminus \{s^*_i\}$ and $s_j \in S_j \setminus \{s^*_j\}$ such that $s_i \succ s_j$. Let $C_\succ$ be the set of all vectors in $E$ that conform to $\succ$, and let $F_\succ = f^{-1}(C_\succ)$. Since $f$ is continuous and $C_\succ$ is closed, $F_\succ$ is closed. Let $\mathcal{F}$ be the collection of nonempty $F_\succ$. Each $\beta \in E$ conforms to some $\succ$, so $\mathcal{F}$ is a closed cover of $X$. Fixing $\succ$ such that $F_\succ$ is nonempty and a particular $i$, it suffices to show that $\bigcap_{\beta \in F_\succ} B_i^X(z, \alpha_i)$ is nonempty. Since $\alpha_i$ is a lower bound on $v_i$, $B_i^X(z, \alpha_i) = \mathcal{X}_i(z)$ for all $z \in X$.

Let $a^1 = (\{s^*_1\}, \{s^*_2\}, \ldots, \{s^*_N\})$. For $1 \leq k \leq |S|$ let $i_k$ be the agent such that the $k$th largest (according to $\succ$) element of $S$ is contained in $S_{i_k}$, and let $s_{i_k}^k$ be this element. For $k > 1$ define $a^k = (a^1_k, \ldots, a^N_k)$ inductively by setting

$$a_{i_k}^k = \begin{cases} a_{i_k}^{k-1} \cup \{s_{i_k}^k\}, & i_k = i, \\ a_{i_k}^{k-1}, & \text{otherwise.} \end{cases}$$
If this is the case and \( k \) is the largest integer such that \( s^k_{i_k} \in \text{argmax}_{s_{i_k} \in S_{i_k}} \beta_{i_k s_{i_k}} \), then
\[
a^k = (\text{argmax}_{i_1 \in S_1} \beta_{1s_1}, \ldots, \text{argmax}_{s_N \in S_N} \beta_{N s_N}).
\]

If \( z \in F_{\infty} \), then \( a(z) = a^k \) for some \( k \). In particular, the \( a(z) \) for the various \( z \in F_{\infty} \) are completely ordered by coordinatewise set inclusion.

One possibility is that \( d_i(a_{-i}(z_{-i})) \notin B_i \) for all \( z \in F_{\infty} \), in which case \( X_i(z) = X_i \) for all \( z \in F_{\infty} \), and for any \( y_i^\succ \in X_i \) we have \( y_i^\succ \in \bigcap_{i \in F_{\infty}} X_i(z) \). Otherwise let \( b_i^\succ = \bigcap d_i(a_{-i}(z_{-i})) \) where the intersection is over all \( z \in F_{\infty} \) such that \( d_i(a_{-i}(z_{-i})) \in B_i \).

We have \( b_i^\succ \in B_i \) because \( B_i \) is closed under intersection. Since the \( a_{-i}(z_{-i}) \) for the various \( z \in F_{\infty} \) are completely ordered by coordinatewise set inclusion, \( \text{(D2)} \) gives a \( a_i^\succ \in A_i \) such that \( a_i^\succ \subset d_i(a_{-i}(z_{-i})) \) for all \( z \in F_{\infty} \), so that \( a_i^\succ \subset b_i^\succ \). Condition (d) of Definition 4.1 now tells us that there is \( y_i^\succ \in Y_{b_i^\succ} \) satisfying \( a_i(y_i^\succ) = a_i^\succ \). If \( z \in F_{\infty} \) and \( d_i(a_{-i}(z_{-i})) \in B_i \), then by (c) of Definition 4.1 we have \( y_i^\succ \in Y_{b_i^\succ} \subset Y_{d_i(a_{-i}(z_{-i}))} \), so \( y_i^\succ \in \bigcap_{i \in F_{\infty}} X_i(z) \) in this case as well. \( \blacksquare \)

We now show that Theorem 5.1 can be used to give a brief proof of Sperner’s Lemma. This is significant because it shows that Theorem 5.1 is not simpler than the fixed point principle and consequently not likely to be provable by more elementary methods.

Let \( E = \{ e_1, \ldots, e_d \} \) be the set of standard unit basis vectors of \( \mathbb{R}^d \). Let \( \Sigma \) be a simplicial complex such that \( \bigcup_{\sigma \in \Sigma} \sigma \) is the convex hull of \( E \), and let \( V = \bigcup_{\sigma \in \Sigma} V_\sigma \). For each \( F \subseteq E \) let \( V_F \) be the set of \( v \in V \) contained in the convex hull of \( F \).

A Sperner labelling is a function \( \lambda : V \to E \) such that \( \lambda(v) \in F \) for all \( F \subseteq E \) and \( v \in V_F \). A simplex \( \sigma \in \Sigma \) is completely labelled if \( \lambda(V_\sigma) = E \). Sperner’s lemma asserts that any Sperner labelling has a completely labelled simplex.

We now define a two player game \((A_1, A_2, v_1, v_2)\) in which \( A_1 \) is the set of all not empty subsets of \( E \) and \( A_2 = \{ V_\sigma : \emptyset \neq \sigma \in \Sigma \} \). Let \( r : E \to E \) be the function given by \( r(e_i) = e_{i+1} \) if \( i < d \) and \( r(e_d) = e_1 \). The payoffs of the players are given by
\[
v_1(a_1, a_2) = \begin{cases} 
1, & \text{if } a_1 \subset r(\lambda(a_2)), \\
0, & \text{otherwise};
\end{cases} \quad \text{and} \quad v_2(a_1, a_2) = \begin{cases} 
1, & \text{if } a_2 \subset V_{a_1}, \\
0, & \text{otherwise}.
\end{cases}
\]

Each player can obtain utility 1 by responding appropriately to a given pure strategy of the other player: \( v_1(r(\lambda(a_2)), a_2) = 1 \) and \( v_2(a_1, a_2) = 1 \) if \( a_2 \) is a singleton subset of \( a_1 \). Therefore \((a_1, a_2)\) is a Nash equilibrium if and only if \( v_1(a_1, a_2) = 1 = v_2(a_1, a_2) \), which is to say that \( a_1 \subset r(\lambda(a_2)) \) and \( a_2 \subset V_{a_1} \). Since \( \lambda \) is a Sperner labelling, the latter condition implies that \( \lambda(a_2) \subset a_1 \). If \( \emptyset \neq F \subseteq E \) and \( F \subset r(F) \), then \( F = E \), so \((a_1, a_2)\) is a Nash equilibrium if and only if \( a_1 = E \) and \( a_2 = V_\sigma \) for some completely labelled simplex \( \sigma \).
Let $B_1 = A_1 \cup \{\emptyset\}$ and $B_2 = \{V_F : F \subset E\}$. Proposition 4.2 implies that $(A_1, B_1)$ and $(A_2, B_2)$ are realizable. Let $d_1(a_2) = r(a_2)$ and $d_2(a_1) = V_{a_1}$. If $(a_1, a_2)$ is not a Nash equilibrium, then either $a_1 \not\subset r(a_2) = d_1(a_2) \in B_1$ or $a_2 \not\subset V_{a_1} = d_2(a_1) \in B_2$. Thus, (D1) is satisfied. For (D2), note that $d_1$ and $d_2$ are never empty and are decreasing in $a_2, a_1$, respectively. Thus, the game has a Nash equilibrium and $\lambda$ has a fully labeled simplex.

6 A Location Model

Schelling (1971, 1972) explores a number of dynamic processes involving location choices by individuals, emphasizing the relationship between individual preferences to be co-located with similar people and segregated outcomes. More generally, the idea of locational externalities goes back at least to Marshall, and there is now a large related literature that continues to be quite active.

For the most part Schelling does not work with configurations that are stable, in the sense that each agent prefers her current location to any available alternative, taking the locations of others as fixed, and does not prove that such configurations exist. Instead he presents simulations in which agents repeatedly relocate to preferred points myopically, insofar as agents do not anticipate the future relocation decisions of others.

This section presents a game on subsets that is similar in spirit, but instead of examining it dynamically, we show that it has a nonempty set of Nash equilibria, which can be interpreted as stable configurations of location choices. This means that the associated dynamic model has configurations that are potential rest points, and which in this sense can be regarded as potential ultimate outcomes.

Each player $i = 1, \ldots, N$ has a subset $U_i \subset \{1, \ldots, N\} \setminus \{i\}$ of other players who she considers undesirable and another subset $D_i \subset \{1, \ldots, N\} \setminus \{i\}$ of other players who are considered desirable. These players have different roles, and it is not necessary to assume that $U_i$ and $D_i$ are disjoint, although that would be consistent with the spirit of the model.

There is a finite set $S$ of social or geographical locations. Player $i$’s strategy set $A_i$ is the set of all nonempty subsets of some nonempty $S_i \subset S$. If player $i$ plays $a_i$, we say that player $i$ frequents each $s_i \in a_i$. We say that $i$ is a denizen of $s_i$ if $a_i = \{s_i\}$. For $s_i \in S_i$ and $a_{-i} \in A_{-i}$ let

$$u_i(s_i, a_{-i}) = |\{j \in U_i : s_i \in a_j\}| \quad \text{and} \quad d_i(s_i, a_{-i}) = |\{j \in D_i : a_j = \{s_i\}\}|$$

be, respectively, the number of undesirables frequenting $s_i$ and the number of desirable denizens of $s_i$, from $i$’s point of view.

If $U_i$ is not empty, then player $i$’s only concern is to avoid frequenting a location which is frequented by more of her undesirables than the number of her desirables.
who are denizens. For such an $i$ let $y_i : S_i \times A_{-i} \to \{0, 1\}$ be given by

$$y_i(s_i, a_{-i}) = \begin{cases} 0, & \text{if } u_i(s_i, a_{-i}) > d_i(s_i, a_{-i}), \\ 1, & \text{otherwise}. \end{cases}$$

If $U_i$ is empty, then player $i$ is averse to frequenting a location that is not frequented by anyone else that she likes, and we define $y_i$ by

$$y_i(s_i, a_{-i}) = \begin{cases} 0, & \text{if } s_i \notin \bigcup_{j \in D_i} a_j, \\ 1, & \text{otherwise}. \end{cases}$$

The payoff of player $i$ at strategy profile $g \in \mathcal{A}$ is

$$v_i(g) = \min_{s_i \in a_i} y_i(s_i, a_{-i}).$$

We have defined a game on subsets $\Gamma = (\mathcal{A}_1, \ldots, \mathcal{A}_N, v_1, \ldots, v_N)$.

For each $i$ let $\mathcal{B}_i = \mathcal{A}_i \cup \{\emptyset\}$. The pair $(\mathcal{A}_i, \mathcal{B}_i)$ is realizable by Corollary 4.3. Let $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$.

**Theorem 6.1.** The game $\Gamma$ has a $\mathcal{B}$-diagnostic, so it has a Nash equilibrium.

**Proof.** For each $i$ define $d_i$ by setting $d_i(a_{-i}) = \arg\max_{s_i \in S_i} y_i(s_i, a_{-i})$. We need to verify (D1) and (D2).

If $a \in \mathcal{A}$ is not a Nash equilibrium, then for some $i$ we have $v_i(a) = 0$ and $v_i(a_i', a_{-i}) = 1$ for some $a_i' \in \mathcal{A}_i$. This implies that for some $s_i \in a_i$ we have $y_i(s_i, a_{-i}) = 0$ and $y_i(\{s_i\}, a_{-i}) = 1$ for all $s_i' \in a_i' \setminus a_i$. Thus $s_i \notin d_i(a_{-i})$, so $a_i$ is not subset of $d_i(a_{-i})$. Since any subset of $S_i$ is an element of $\mathcal{B}_i$ it follows that (D1) holds.

Let $a_1, a_2, \ldots, a_K$ be a decreasing sequence in $\mathcal{A}$. Fix a player $i$, and note that for any $s_i$, $u_i(s_i, a_{-i}^{K-1})$ is a weakly decreasing function of $k$ and $d_i(s_i, a_{-i}^{K-1})$ is a weakly increasing function of $k$.

First suppose that $U_i$ is not empty. If $u_i(s_i, a_{-i}^{k-1}) > d_i(s_i, a_{-i}^{K-1})$ for all $s_i \in S_i$ and all $k$, then for any $s_i^*$ we have $s_i^* \in d_i(a_1^{K-1}) \cap \cdots \cap d_i(a_K^{K-1})$. Otherwise let $k^*$ be the first number such that $u_i(s_i^{K^*}, a_{-i}^{K-1}) \leq d_i(s_i^{K^*}, a_{-i}^{K-1})$ for some $s_i^{K^*} \in S_i$. Then $u_i(s_i^{K^*}, a_{-i}^{K-1}) \leq d_i(s_i^k, a_{-i}^{K-1})$ and consequently $y_i(\{s_i^k\}, a_{-i}^{K-1}) = 1$ for all $k \geq k^*$. If $k < k^*$, then $u_i(s_i, a_{-i}^{K^*}) > d_i(s_i, a_{-i}^{K-1})$ and $y_i(s_i, a_{-i}^{k}) = 0$ for all $s_i \in S_i$, so that $s_i \notin d_i(a_{-i}^{K-1})$ for all $s_i \in S_i$. In particular, for all $k$ we have $s_i^* \in d_i(a_{-i}^{K-1})$. Letting $a_i^* = \{s_i^*\}$ we conclude that (D2) is satisfied.

Suppose now that $U_i$ is empty. If $a_j^1 \cap S_i = \emptyset$ for all $j \neq i$, then for any $s_i^* \in S_i$ we have $s_i^* \in d_i(a_{-i}^{K-1})$ for all $k$. Otherwise let $k^*$ be the largest integer such that $a_j^1 \cap S_i \neq \emptyset$ for some $j$, and choose $s_i^*$ in such an intersection. Again we have $s_i^* \in d_i(a_{-i}^{K-1})$ for all $k$. We conclude that (D2) is also satisfied in this case. ■
Insofar as we wish to convincingly illustrate the usefulness of our results, it is important to consider whether other methods could be used to show that a game on subsets has a Nash equilibrium. The following particular instance of the location model illustrates how various other methods are inapplicable.

There are two players and three locations. Let \( S_1 = \{a, b, c\} \) and \( S_2 = \{a, b\} \). Player 1 considers Player 2 desirable while Player 2 considers Player 1 undesirable, so that \( U_1 = \emptyset \), \( D_1 = \{2\} \), \( U_2 = \{1\} \), and \( D_2 = \emptyset \). Therefore the payoffs are as follows:

\[
\begin{array}{ccc}
P2 & \{a\} & \{b\} & \{a, b\} \\
\{a\} & (1, 0) & (0, 1) & (1, 0) \\
\{b\} & (0, 1) & (1, 0) & (1, 0) \\
\{c\} & (0, 0) & (0, 0) & (0, 1) \\
P1 & \{a, b\} & \{a\} & \{b\} \\
\{a, b\} & (0, 0) & (0, 0) & (0, 0) \\
\{a\} & (0, 0) & (0, 0) & (0, 0) \\
\{b, c\} & (0, 0) & (0, 0) & (0, 0) \\
\{a, b, c\} & (0, 0) & (0, 0) & (0, 0) \\
\end{array}
\]

One can easily check that \( (\{a, b\}, \{a, b\}) \) is the only pure strategy Nash equilibrium.

There is a mixed Nash equilibrium \( (\frac{1}{2}\{a\} + \frac{1}{2}\{b\}, \frac{1}{2}\{a\} + \frac{1}{2}\{b\}) \) that is not a pure equilibrium. Therefore one cannot prove existence of a pure Nash equilibrium by combining the existence of a mixed equilibrium with an argument showing that all equilibria are pure.

A strategic form game is a \textit{potential game} (Rosenthal (1973); Monderer and Shapley (1996)) if there is a real valued \textit{potential function} on the set of pure strategy profiles such that for any pair of profiles that differ in only one agent’s component, the difference in that agent’s utility at the two profiles has the same sign as the difference in the potential function at the two profiles. Any maximizer of the potential function is a Nash equilibrium, so Nash equilibria exist when the pure strategy sets are finite. For any potential game, any game obtained by restricting each agent to a subset of her strategies is also a potential game. The game obtained by restricting each agent to play either \( \{a\} \) or \( \{b\} \) is an instance of matching pennies, which of course has no Nash equilibrium and is thus \textit{not} a potential game.

A strategic form game exhibits \textit{strategic complementarities} or \textit{increasing differences} (Bulow et al. (1985)) if the agents’ sets of pure strategies are partially ordered and increasing one agent’s strategy, relative to the given ordering, weakly increases the desirability for the other agents of increasing their strategies. If various auxiliary conditions are satisfied, a game that exhibits strategic complementarities necessarily has Nash equilibria. In a game on subsets the pure strategies are ordered by both inclusion and reverse inclusion. However, we have

\[ v_1(\{a, b\}, \{a, b\}) - v_1(\{a\}, \{a, b\}) > v_1(\{a, b\}, \{a\}) - v_1(\{a\}, \{a\}) \]
and
\[ v_1(\{a, b, c\}, \{a, b\}) - v_1(\{a, b\}, \{a, b\}) < v_1(\{a, b, c\}, \{a\}) - v_1(\{a, b\}, \{a\}), \]

so the location game does not exhibit strategic complementarities.

Consider the game \( G = (X_1, X_2, u_1, u_2) \) from the proof of Theorem 5.1 and the setting of Corollary 4.3 where \( X_1 = \Delta^2 \) in \( \mathbb{R}^3 \) and \( X_2 = \Delta^1 \) in \( \mathbb{R}^2 \), and a generic point in \( X \) is \( x = (x_1, x_2) = ((x_{1a}, x_{1b}, x_{1c}), (x_{2a}, x_{2b})) \). The functions \( f_1 : X_1 \to \mathbb{R}^3 \) and \( f_2 : X_2 \to \mathbb{R}^2 \) are the respective identity functions, so
\[ u_i(x) = v_i(\text{argmax}_{s \in \{a, b, c\}} x_{1s}, \text{argmax}_{t \in \{a, b\}} x_{2t}) \]
for both \( i \). The point \((1, 0, 0), (1/2, 1/2)\) is not a Nash equilibrium, and it is not hard to show that the game is not better reply secure at this point, so Theorem 6.1 cannot be proved by applying Reny’s theorem to the games arising in the proof of Theorem 5.1.

It can happen that the a game has an upper semicontinuous best reply correspondence even though the payoff functions are discontinuous, but one can see that this is not the case for \( G \) by considering a sequence \( x_1^n \) which correspond to Player 1 playing \( \{a\} \) converging to \( x_1 \) that corresponds to \( \{a, b\} \). Noting that Player 2’s best response to Player 1 playing \( x_1 \) does not include \( \{a, b\} \). Therefore, the best response correspondence is not upper hemicontinuous.

### 7 The Proof of Theorem 3.4

We say that a (not necessarily Hausdorff) topological space is regular if each point has a neighborhood base of closed sets. (This is the definition given by Kelley (1955), which differs from the usage of some other authors.) Topological vector spaces are regular topological spaces, even if they are not Hausdorff (e.g., Schaefer (1971, p. 16)). It is easy to see that any subspace of a regular space is regular, and that finite cartesian products of regular spaces are regular, so each \( X_i \), each \( X_{-i} \), and \( X \) are all regular.

In preparation for the main body of the argument we present two lemmas, the first of which is a variant of the fixed point principle that holds in topological vector spaces that are neither Hausdorff nor locally convex. Although it bears some resemblance to the Knaster-Kuratowski-Mazurkiewicz lemma, we have not managed to forge a direct connection between the two results.

**Lemma 7.1.** Let \( X \) be a nonempty compact convex subset of a topological vector space \( Y \) and let \( P : X \to X \) be a set valued mapping. If there is a finite closed cover \( H_1, \ldots, H_L \) of \( X \) such that \( \bigcap_{z \in H_j} P(z) \neq \emptyset \) for each \( j = 1, \ldots, L \), then there exists \( x^* \in X \) such that \( x^* \in \text{con} \ P(x^*) \).
Proof. For each \( j = 1, \ldots, L \) choose a \( y_j \in \bigcap_{i \in H_j} P(z) \). Let \( \mathbf{e}_1, \ldots, \mathbf{e}_L \) be the standard unit basis vectors of \( \mathbb{R}^L \), let \( \Delta \) be their convex hull, and let \( \pi : \Delta \to X \) be the map \( \omega \mapsto \sum_j \omega_j y_j \); this is continuous because addition and scalar multiplication are continuous operations in any topological vector space. Define a set valued mapping \( Q : \Delta \to \Delta \) by letting \( Q(\omega) \) be the convex hull of \( \{ \mathbf{e}_j : \pi(\omega) \in H_j \} \). This is nonempty valued because the \( H_j \) cover \( X \), it is upper semicontinuous because each \( H_j \) is closed, and of course it is convex valued, so Kakutani’s fixed point theorem implies that it has a fixed point \( \omega^* \). Let \( x^* = \pi(\omega^*) \). Then

\[
x^* \in \text{con} \{ y_j : x^* \in H_j \} \subset \text{con} \left( \bigcup_{j : x^* \in H_j} \left( \bigcap_{x \in H_j} P(x) \right) \right) \subset P(x^*).
\]

\[\Box\]

Lemma 7.2. Suppose that \( \alpha_1, \ldots, \alpha_\ell \in \mathbb{R}^N \), \( U_1, \ldots, U_\ell \) are open subsets of \( X \), and, for each \( h = 1, \ldots, \ell \), the game is multiply \( (\mathcal{X}, \alpha_h) \)-secure on \( U_h \). Let \( \alpha = \max_{h=1}^\ell \alpha_h \) and \( U = \bigcap_{h=1}^\ell U^h \). Then the game is multiply \( (\mathcal{X}, \alpha) \)-secure on \( U \).

Proof. For each \( h \) there is a cover \( F^1_h, \ldots, F^j_h \) of \( U_h \) by relatively closed sets such that for each \( i \) and \( j \) we have \( \bigcap_{i \in F_j^h} B_i^X(z, \alpha^h_i) \neq \emptyset \). This condition continues to hold with \( U_h \) replaced by \( U \). It also continues to hold if \( F^1_h, \ldots, F^{j'}_h \) is replaced by the collection \( G^1, \ldots, G^J \) of all nonempty intersections of the form \( F^1_i \cap \ldots \cap F^j_h \). Then \( \bigcap_{i \in G_j} B_i^X(z, \alpha^h_i) \neq \emptyset \) for all \( i = 1, \ldots, N, j = 1, \ldots, J, \) and \( h = 1, \ldots, \ell \). For each \( i \) there is \( h \) such that \( \alpha_i = \alpha^h_i \), so for any \( j \) we actually have \( \bigcap_{i \in G_j} B_i^X(z, \alpha_i) \neq \emptyset \), so \( j \) can multiply \( \mathcal{X} \)-secure \( \alpha_i \) at \( z \).

For any \( z \in U \) there are \( h \) and \( i \) such that \( z_i \notin C^X(z, \alpha^h_i) \). Since \( \alpha_i \geq \alpha^h_i \), this implies that \( z_i \notin C^X(z, \alpha_i) \). \( \Box \)

We now have the tools we need to complete the proof of Theorem 3.4. Aiming at a contradiction, suppose that \( G \) is MR secure but has no Nash equilibrium. Then there is a restriction operator \( \mathcal{X} \) such that \( G \) can be covered by open sets on which \( G \) is multiply \( \mathcal{X} \)-secure. Since \( X \) is regular, each \( x \in X \) is contained in a closed set that is contained in one of these open sets, and since \( X \) is compact there is a finite collection of such closed sets whose interiors cover \( X \). Therefore there are closed sets \( F_1, \ldots, F_m \) whose interiors cover \( X \) and \( \alpha_1, \ldots, \alpha_m \in \mathbb{R}^N \) such that for each \( h \) the game is multiply \( (\mathcal{X}, \alpha_h) \)-secure on a neighborhood of \( F_h \). Define \( \psi \) by setting

\[
\psi(x) = \max_{x \in F_h} \alpha_h.
\]

For each \( i \) and \( x \) let \( P_i(x) = B_i^X(x, \psi_i(x)) \). For each \( x \) let \( P(x) = P_1(x) \times \cdots \times P_N(x) \).

Consider a particular agent \( i \) and \( x \in X \). By Lemma 7.2 the game is multiply \( (\mathcal{X}, \psi(x)) \)-secure on some neighborhood \( U \) of \( x \), so there is a cover \( G_1, \ldots, G_J \) by relatively closed sets such that \( \bigcap_{i \in G_j} B_i^X(z, \psi_i(x)) \neq \emptyset \) for each \( j \). Since \( X \) is
regular, $U$ contains a closed neighborhood $C$. Since $\psi_i$ is upper semicontinuous and takes on finitely many values, if $C$ is small enough, then $\psi_i(z) \leq \psi_i(x)$ for all $z \in C$. After replacing each $G_j$ with $G_j \cap C$ we have

$$\bigcap_{z \in G_j} P_i(z) = \bigcap_{z \in G_j} B^X_i(z, \psi_i(z)) \supseteq \bigcap_{z \in G_j} B^X_i(z, \psi_i(x)) \neq \emptyset.$$  

Since $X$ is compact, for each $i$ it has a finite cover $G_{i1}, \ldots, G_{iK_i}$ by closed sets with this property. If $H_1, \ldots, H_L$ are the nonempty intersections of the form $G_{1k_1} \cap \ldots \cap G_{N k_N}$, then these sets are a closed cover of $X$ and $\bigcap_{z \in H_i} P(z) \neq \emptyset$ for each $l$. Thus the hypotheses of Lemma 7.1 are satisfied by $P: X \to X$, so there is an $x^* \in X$ satisfying $x^* \in \text{co} P(x^*)$, which is to say that $x^*_i \in C^X_i(x^*, \psi_i(x^*))$ for all $i$. But the game is multiply $(\mathcal{X}, \psi(x^*))$-secure at $x^*$, so for some $i$ we have $x^*_i \notin C^X_i(x^*, \psi_i(x^*))$. This contradiction completes the proof.

8 Payoff Secure Games

Reny points out that a combination of two conditions, payoff security and reciprocal upper semicontinuity, imply the hypotheses of his existence result, and in applications it is typically relatively easy to verify them when they hold. We now define generalizations of these notions in our setting, and establish that together they imply that the game is restrictionally secure.

Reny’s notion of payoff security requires that for each $x$ and $\varepsilon > 0$ each player $i$ can secure $u_i(x) - \varepsilon$ at $x$.

**Definition 8.1.** The game $G$ is multiply $\mathcal{X}$-payoff secure at $x \in X$ if, for each $\varepsilon > 0$, each player $i$ can multiply $\mathcal{X}$-secure $u_i(x) - \varepsilon$ at $x$. The game $G$ is multiply $\mathcal{X}$-payoff secure if it is $\mathcal{X}$-secure at each $x \in X$.

Reny’s reciprocal upper semicontinuity (which was introduced by Simon (1987) under the name “complementary discontinuities,” and which generalizes the requirement in Dasgupta and Maskin (1986) that the sum of payoffs be upper semicontinuous) requires that $u(x) = \alpha$ whenever $(x, \alpha)$ is in the closure of the graph of $u$ and $\alpha \geq u(x)$. Equivalently, for each possible “jump up” $\delta > 0$ there is a neighborhood $U$ of $x$ and a “jump down” $\varepsilon > 0$ such that for each $z \in U$, if $u_i(z) > u_i(x) + \delta$ for some $i$, then there must be a player $j$ with $u_j(z) < u_j(x) - \varepsilon$.

**Definition 8.2.** The game $G$ is $\mathcal{X}$-reciprocally upper semicontinuous at $x \in X$ if for every $\delta > 0$ there exists $\varepsilon > 0$ and a neighborhood $U$ of $x$ such that for each $z \in U$, if there is a player $i$ with $z_i \in C^X_i(z, u_i(x) + \delta)$, then there is a player $j$ such that $z_j \notin C^X_j(z, u_j(x) - \varepsilon)$. The game $G$ is $\mathcal{X}$-reciprocally upper semicontinuous if it is $\mathcal{X}$-reciprocally upper semicontinuous at each $x \in X$. 
The next result is the analogue in our setting of Reny’s Proposition 3.2, which asserts that payoff security and reciprocal upper semicontinuity imply better reply security. In conjunction with Theorem 3.4 it implies a generalization of Reny’s Corollary 3.3. We say that $X$ is other-directed if, for each $i$, $X_i(x)$ depends only on $x_{-i}$, so that $X_i(y_i, x_{-i}) = X_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $y_i, z_i \in X_i$.

**Proposition 8.3.** If $X$ is other-directed and $G$ is multiply $X$-payoff secure and $X$-reciprocally upper semicontinuous, then it is multiply $X$-secure.

**Proof.** Fixing an $x \in X$ that is not a Nash equilibrium, our goal is to show that the game is multiply $X$-secure at $x$. Let $\mathcal{I}$ be the set of players such that $x_i$ is not a best response to $x_{-i}$. For each $i \in \mathcal{I}$, choose $y_i$ such that $u_i(y_i, x_{-i}) > u_i(x)$, and let $\delta > 0$ be small enough that $u_i(y_i, x_{-i}) > u_i(x) + \delta$ for all $i \in \mathcal{I}$. Since the game is multiply $X$-payoff secure, each $i \in \mathcal{I}$ can multiply $X$-secure $\alpha_i = u_i(x) + \delta$ at $(y_i, x_{-i})$. Since $X$ is other-directed, player $i$ can also multiply $X$-secure $\alpha_i$ at $x$.

Since the game is $X$-reciprocally upper semicontinuous, there is an $\varepsilon > 0$ and a neighborhood $U$ of $x$ such that for all $z \in U$, if $z_i \in C^X_i(z, u_i(x) + \delta)$ for some $i$, then $z_j \notin C^X_j(z, u_j(x) - \varepsilon)$ for some $j$. Since the game is multiply $X$-payoff secure, each $j \notin \mathcal{I}$ can multiply $X$-secure $\alpha_j = u_j(x) - \varepsilon$ at $x$. It is now the case that for each $z \in U$, either $z_i \notin C^X_i(z, \alpha_i)$ for all $i \in \mathcal{I}$ or $z_j \notin C^X_j(z, \alpha_j)$ for some $j \notin \mathcal{I}$. 

Reny points out (Corollary 3.4) that if each $u_i$ is lower semicontinuous in the strategies of the other players, then the game is necessarily payoff secure because for each $x$, $\varepsilon > 0$, and $i$, player $i$ can use $x_i$ to secure $u_i(x) - \varepsilon$.

**Definition 8.4.** We say that $u_i$ is multiply lower semicontinuous in the strategies of the other players if for every $x \in X$ there exists a finite closed cover $F^1, \ldots, F^J$ of a neighborhood of $x$ such that for each $j$ there is some $y^j_i \in X_i$ satisfying $u_i(y^j_i, x_{-i}) \geq u_i(x)$ and the function $z_{-i} \mapsto u_i(y^j_i, z_{-i})$ restricted to $F^j$ is lower semicontinuous.

The following proposition is a consequence of Proposition 8.3.

**Corollary 8.5.** If each $u_i$ is multiply lower semicontinuous in the strategies of other players, then it is multiply payoff secure. If in addition for each $i$ and each $x_{-i} \in X_{-i}$ the function $u_i(\cdot, x_{-i}) : X_i \rightarrow \mathbb{R}$ is quasiconcave and the game is reciprocally upper semicontinuous, then there exists a a pure strategy Nash equilibrium.

Bagh and Jofre (2006) show that better reply security is implied by payoff security and a condition that is weaker than reciprocal upper semicontinuity. We have not found a suitable analog of that concept. Carmona (2009) shows that existence of equilibrium is implied by a weakening of payoff security and a weak form of upper semicontinuity. We do not know of analogs of these conditions in our setting.
9 Conclusion

We have weakened the hypotheses of Reny’s theorem in several directions, achieved a briefer\(^2\) proof, and also extended his notions of payoff security and reciprocal upper semicontinuity. The generalized theorem has been used to prove the existence of pure Nash equilibrium in a new class of finite games, games on subsets, which includes examples in the spirit of locations models studied by Schelling (1971, 1972). Our result implies the Nishimura and Friedman (1981) existence theorem, which was one of the two earlier results not encompassed by Reny’s theorem, the other being Cournot equilibrium with fixed costs.

The concepts introduced in Section 3 were inspired in part by the hope of achieving a result that could be used to prove the existence result of Novshek (1985), which is the most refined result asserting existence of Cournot equilibrium in the stream of literature following McManus (1962, 1964), Roberts and Sonnenschein (1976), and Szidarovzky and Yakowitz (1977). Specifically, it will often happen that the set of quantities that allow a firm to obtain at least a certain level of profits will not be convex because it will include both increases and reductions to zero. Introducing a restriction operator allows the analyst to restrict attention to one possibility or the other at each point.

A proof of Novshek’s result applying our result can be accomplished for the case of two firms, and was presented in an earlier version of this paper, but extending the argument to an arbitrary number of firms proved to be quite difficult. This seems to be related to the fact that although the general case is a game of strategic substitutes, in the sense of Bulow et al. (1985), the case of two firms can be construed as a game of strategic complements if the ordering of one of the two firm’s sets of quantities is reversed.

Subsequent literature related to Novshek’s result has gone in a different direction. Kukushkin (1994) presents a brief proof of the general case. As with various proofs of topological fixed point theorems, Kukushkin’s argument combines a non-trivial combinatoric result with a straightforward limiting argument. Kukushkin (2004) gives a more general treatment of finite games in which there is an ordering of strategy profiles that is increased by best response dynamics. Huang (2002), Dubey et al. (2006), and Kukushkin (2007) develop an approach to games of strategic substitutes and complements that generalizes the notion of a potential game. Jensen (2010) introduces a class of games that allows the methods of all these authors to be applied more broadly. In all of this literature a key feature is the presence of useful univariate aggregators.

There are many issues related to games on subsets that could be pursued. For example we do not know of an algorithm for computing a Nash equilibrium other

---

\(^2\)Prokopovych (2010b) gives a nice short proof for quasiconcave better reply secure games whose sets of pure strategies are metric spaces.
than exhaustive search. Since the size of the set of pure strategy profiles for the location model is an exponential function of the data required to define an instance, there is the question of whether there is a polynomial time algorithm for computing an equilibrium, and, if not, what can be said about the computational complexity of this problem. The class of games on subsets is evidently quite rich, and might have a number of applications that are not apparent at present. Even considering only the location model, an examination of our methods quickly reveals that the set up could be varied in many directions.

Bibliography


