I. Introduction.

A. General equilibrium theory explains economic outcomes in terms of rational behavior in a market environment.
   1. The model does not explain the emergence of markets or the price formation process.
   2. Our intuition is that these aspects of economic equilibrium will emerge as a consequence of competition in an underlying game.

B. Desirable features of game theoretic models.
   1. The postulated game should be “realistic”.
      a. Of course the actual world is very complicated, and what this really means is that the model should not be unrealistic in a crucial way.
      b. In particular, the model should not “prohibit” natural forms of competition that would change the set of equilibrium outcomes.
   2. The model should encompass a large class of games.
      a. Our intuition is that competition is not an artifact of special features of the actual situation, but would be the outcome in the absence of powerful forces in some other direction.
      b. Here it is the generality of the argument, rather than the particular class of games, that is crucial.

C. The Axiomatic Approach.
   1. One describes a class of games axiomatically and shows that all equilibria are Walrasian.

3. Related to this is the general discussion of continuity in the limit (as the number of agents becomes large) of the equilibrium correspondence – see Green, “Continuum and Finite Player Noncooperative Models of Competition,” *Econometrica* 1984.

D. Concrete Models.

1. The Shapley-Shubik game is perhaps the best known market game.
   a. There is a market for each good, and in each market each agent submits a non-negative offer – an amount of the good – and a bid – a sum of money.
   b. The price in the market is
      \[
      \frac{(\text{total bid})}{(\text{total offer})}.\]
   c. The total offer is divided among the bidders in proportion to their bids, and the total revenue is divided among the suppliers in proportion to their offers.
   d. Typically there are a continuum of equilibrium allocations, many of them non-Walrasian.
   e. Note that private mutually beneficial exchange outside the organized markets is prohibited. That is, the rules make no provision for it.

2. The model we examine closely was first posed by Rubinstein and Wolinsky.
   a. This is a model of sequential bargaining between agents who are randomly paired in each period.
      i. In each period agents are paired randomly.
      ii. In each pair the “proposer” is designated, randomly, and he proposes a net trade.
      iii. The other agent is the “responder,” and he either accepts or rejects the offer.
iv. One’s payoff is the utility of the bundle of goods with which one leaves
the market. One can leave any time, and there is no penalty for leaving
later.

b. Rubinstein and Wolinsky examine this model in an economy with two goods,
cars and cash, one of which is indivisible.

c. Gale proved that a version of this game, in an exchange economy, has exactly
the Walrasian allocations as its set of subgame perfect equilibria.

d. Our treatment differs from Gale’s in two ways.

i. Our argument is based on a characterization of Walrasian outcomes that
clearly displays the forces leading to a competitive outcome.

ii. Our mathematical assumption are also somewhat different.

II. The Description of Continuum Economies

A. The consumption set for all agents is $\mathcal{X} = \mathbb{R}^\mathcal{E}_{++}$.

B. Preferences are represented by utility functions $u : \mathcal{X} \to \mathbb{R}$ that are strictly
increasing, bounded, $C^1$, and whose indifference curves are closed in $\mathbb{R}^\mathcal{E}$.

1. Let $\mathcal{U}$ be the set of utility functions with these properties.

a. We endow this space with the topology of uniform $C^1$ convergence on
compacta.

b. This topology is metrizable and separable.

C. The space of characteristics is $\mathcal{C} = \mathcal{U} \times \mathbb{R}^\mathcal{E}$ with typical element $c = (u_c, x_c)$.

1. The assumption that unlimited short sales are permitted during play makes
the proof easier, but the theorems do not depend on this as Gale shows.

2. It is conceptually important that agents do not observe their partners’ histories
of play or noneconomic characteristics.

a. Part of the appeal of the market system is that people of similar resources
should be able to obtain similar outcomes.
i. The idea is that competition prevents discrimination on any basis other than ability to pay.

ii. There are examples of equilibria whose allocations are non–Walrasian if people can observe their partners’ histories.

3. Auxiliary Notation.
   a. Let \( \zeta : \{(c, p) \in C \times \mathbb{R}_+^{|p|} | p \cdot x_c > 0 \} \rightarrow \mathbb{R}_+^{|p|} \) be the excess demand correspondence.
   b. For \( Z \subset \mathbb{R}_+^{|p|}, c \in C, \) let
      \[
      V(c, Z) = \begin{cases} 
      \sup\{u_c(x_c + z) | z \in Z \} & \text{if} \ (x_c + Z) \cap X \neq \emptyset \\
      -\infty & \text{otherwise}
      \end{cases}
      \]

D. Economies

1. Definition: An economy is a finite positive Borel measure \( \lambda \) on \( C \) such that \( \int x \, d\lambda \) is defined.

2. An allocation for \( \lambda \) is an economy \( \nu \) such that
   a. The marginal distribution of \( \lambda \) and \( \nu \) or \( C \) coincide.
   b. \( \int x \, d\lambda = \int x \, d\nu. \)
   c. \( x_c \in X \) for \( \nu \)-almost all \( c \in C \), i.e. \( \nu(U \times X) = \nu(C). \)

3. Remark: This is somewhat unconventional in that there is no underlying space of agents.
   a. Although final outcomes turn out to be deterministic, an agent’s progress through the game is a stochastic process.
   b. Keeping track of individuals’ bundles would require random functions satisfying the law of large numbers almost surely.
   c. Work by Judd, Feldman and Gilles, and Ed Green shows that there are severe mathematical difficulties with the notion of a continuum of i.i.d. random variables obeying the law of large numbers.
   d. We avoid these issues by discussing only population averages.
III. Theorem 1.

A. We characterize allocations that result when agents choose from a set of net trades of the type that, intuitively, are available in an equilibrium of a Gale type game.

B. Theorem 1: Suppose $\nu$ is an allocation for $\lambda$, and there is a set $Z \subset \mathbb{R}^d$ B such that:

(i) $0 \in Z = Z + Z$;

(ii) if $c \in \text{supp} \ \nu$ and $x_c \in X$, then $z \in Z$ for any $z \in \mathbb{R}^d$ such that $u_c(x_c - z) > u_c(x_c)$;

(iii) for all Borel sets $E \subset U$ and all $\bar{u} \in \mathbb{R}$,

$$\lambda(\{c|u_c \in E \text{ and } V(c, Z) \geq \bar{u}\}) \leq \lambda(\{c|u_c \in E \text{ and } u_c(x_c) \geq \bar{u}\}).$$

Then $\nu$ is Walrasian for $\lambda$ : there is a price vector $p \in \mathbb{R}^d_+ \neq 0$ such that

($\alpha$) $\lambda(\{c|p \cdot x_c \leq 0\}) = 0$;

($\beta$) $\nu(\{c \in U \times X|0 \in \zeta(c, p)\}) = \nu(C)$;

($\gamma$) for all Borel sets $E \subset U$ and all $w \in \mathbb{R}$,

$$\lambda(\{c|p \cdot x_c \leq w\}) = \nu(\{c|p \cdot x_c \leq w\}).$$

C. In a traditional formulation an economy is a measurable function $E; (\Omega, A, M) \to C$ from a measure space to the space of characteristics.

1. In this approach an allocation is a measurable function $f : \Omega \to X$.

D. Proof of Theorem 1.

1. (i), (ii) and the differentiability of utilities imply that there is $p \in \mathbb{R}_+^d$ such that

$$\{z|p \cdot z < 0\} \subset Z \subset \{z|p \cdot z \leq 0\}.$$ 

2. The problem now reduces to comparing (iii) and the assertion of Theorem 1.

a. Special Case– supp $\lambda \subset \{u\} \times \mathbb{R}^d$. 

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b. (iii) asserts that the distribution of utility levels in $\nu$ “first order stochastic
dominates” the distribution that would result from preference maximization
at prices $p$.
c. Thus the distribution of wealths (relative to $p$) in $\nu$ must F.O.S.D. the
distribution of wealths in $\lambda$.
d. Since the total quantity of goods are the same, the two distribution of
wealths must be the same, and goods must be distributed efficiently in $\nu$.
e. The general result is proved by “integrating” this argument over all utility
functions.

IV. The Market Game.
A. Broad Description.
1. There is a population in the market prior to the first period.
2. In each period a population enters.
3. The sum is the population in the market after entry.
   a. Paired randomly.
   b. One agent in each pair is the proposer, the other is the responder.
   c. The proposer proposes a net trade.
   d. The responder accepts the trade, declines, or leaves.
4. Payoffs.
   a. One receives the utility of the bundle one leaves with.
   b. One must leave in finite time to avoid a payoff of $-\infty$, but there is no
      penalty for staying in a long time.
B. Formal description.
1. $\lambda_t$ – flow of new agents into the market in period $t$.
2. $\mu_t$ – population in the market after entry in period $t$. 
3. \( \frac{\mu_t}{\mu_t(C)} \) is the probability distribution over potential partners in period \( t \).

V. Behavior Strategies

A. A bit of measure theory.

1. If \((\Omega, \mathcal{A})\) is a measurable space, let \( \mathcal{M}(\Omega) \) be the set of strictly positive finite measures on \( \Omega \), and let \( \mathcal{P}(\Omega) \) be the set of probability measures on \( \Omega \).

2. A function \( P_{12} : \Omega_1 \to \mathcal{P}(\Omega_2) \) is a transition probability if \( P_{12}(\cdot)(A_2) : \Omega_1 \to [0, 1] \) is measurable for all \( A_2 \in \mathcal{A}_2 \). For each \( M \in \mathcal{M}(\Omega_1) \) there is a unique measure \( M \otimes P_{12} \in \mathcal{M}(\Omega_1 \times \Omega_2) \) satisfying

\[
(M \otimes P_{12})(A_1 \times A_2) = \int_{A_1} P_{12}(\cdot)(A_2) \, dM.
\]

a. Let \( \mathcal{P}(\Omega_1, \Omega_2) \) be the set of transition probabilities \( P_{12} : \Omega_1 \to \mathcal{P}(\Omega_2) \).

b. \( M \otimes P_{12} \) is the marginal on \( \Omega_2 \).

B. The strategies of proposers and responders in period \( t \) are

\[
P_t \in \mathcal{P}(C, \mathbb{R}^d) \quad \text{and} \quad R_t \in \mathcal{P}(C \times \mathbb{R}^d, \{A, D, L\}).
\]

C. The distribution of outcomes of pairings in period \( t \) is

\[
\xi_t = (((\mu_t \otimes P_t) \times \mu_t) \times R_t) \in \mathcal{P}(C \times \mathbb{R}^d \times C \times \{A, D, L\}).
\]

1. This induces a distribution of characteristics leaving the market in period \( t \) that we denote by \( \nu_t \).

VI. Equilibrium

A. We give only an informal description.

1. An individual in this system faces a dynamic programming problem.

2. A policy for an agent is essentially a sequence of transition probabilities like

\[
((P_0, R_0), (P_1, R_1), \ldots).
\]
3. The value of such a policy can be computed.
   a. Compute distribution over histories.
   b. Payoffs are measurable function of histories.

B. An equilibrium is a situation in which, for all characteristics and all dates,
   \(((P_0, R_0), (P_1, R_1), \ldots)\) is optimal.

VII. Consequences of Equilibrium

A. We define a value \(W_t(c)\) of a characteristic \(c\) in period \(t\) prior to pairing.

B. **Assumption A:** Everybody can achieve a strictly positive consumption bundle with
   probability 1.

C. Agents will be leaving indefinitely far into the future.

D. You can always wait a period.

E. If a person is about to leave they accept any improving offer.

F. There is a single normalized price that is the supporting price of \(u_c\) at \(x_c\) for all
   \(c \in \text{supp } \nu_1\).

   1. Otherwise people who are leaving at one of the prices should stick around.

VIII. Law of Single Price.

A. It appears that one has to have a compact support for \(\{\nu_1\}\).

   1. Example.

B. If one does have effectively compact support then one can be sure of encountering
   people down the road who are about to leave, provided enough people leave.

C. Thus \(W_t(c) \geq u_c(x_c + \zeta(c, p))\).

   1. In fact equality, since there cannot be a last period in which it is possible to
      get a positive value net trade.
IX. There are two cases in which we get Walrasian outcomes.

A. Necessity of having two economies to compare.

B. Case 1: \( \lambda_\infty = \Sigma_t \lambda_t \) and \( \nu_\infty = \Sigma_t \nu_t \) are both finite.

   (a) \( \lambda_\infty(C) = \nu_\infty(C) \).

   (b) \( \int x \, d\lambda_\infty = \int x \, d\nu_\infty \).

   (c) Assumption A.

   (d) \( \text{supp} \, \nu_\infty \subseteq \mathcal{U} \times \mathcal{X} \) is compact. (This is unfortunate and possibly unnecessary.)

C. Case 2:

\[
\frac{\sum_{t=1}^{T} \lambda_t}{T} \rightarrow \lambda_M, \quad \frac{\sum_{t=1}^{T} \nu_t}{T} \rightarrow \nu_M.
\]

   (a) all endowments positive.

   (b) \( \lambda_M(C) = \nu_M(C) \).

   (c) \( \int x \, d\lambda_M = \int x \, d\nu_M \).

D. Theorem 2: If the assumptions of either case are satisfied then \( \nu_\infty \) (or \( \nu_M \)) is Walrasian for \( \lambda_\infty \) (or \( \nu_M \)).