Index Theory

I. Introduction

A. Over a fifty year period, fixed point theory became more refined in several senses.

1. Brouwer’s fixed point theorem was generalized from functions to convex valued correspondences, and then to contractible (actually “acyclic”) valued correspondences.

2. Brouwer’s fixed point theorem was generalized from finite dimensional sets to compact convex subsets of Banach spaces, then to compact convex subsets of locally convex topological vector spaces, and finally to compact absolute retracts.

   a. A topological vector space is a vector space with a Hausdorff topology with respect to which addition and scalar multiplication are continuous.

   b. Since the topology is translation invariant, it is determined by a neighborhood base at the origin.

   c. The space is locally convex if every neighborhood of the origin contains a convex neighborhood of the origin.

3. The theory has been refined to distinguish between essential and inessential fixed points, and between different sorts of essential fixed points, and these distinctions have been extended to sets of fixed points.

B. All of these developments are summarized by the main theorem of index theory.

1. To understand the topological generality we need concepts from the theory of retracts.

2. Otherwise the index axioms are easily stated.
C. After stating this theorem we will explain some of its consequences.
D. As time permits, we will explain some of the main ideas of the proof.

II. Retracts
A. Let $X$ be a metric space, and let $A$ be a subset of $X$.
   1. We say that $A$ is a retract of $X$ if there is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$.
   2. Such an $r$ is called a retraction.
B. Many desirable properties that $X$ might have are necessarily also possessed by the retract $A$.
   1. A topological space $X$ has the fixed point property if every continuous function $f : X \to X$ has a fixed point.

Lemma: If $X$ has the fixed point property and $A$ is a retract of $X$, then $A$ has the fixed point property.

Proof: Let $r : X \to A$ be a retraction. If $f : A \to A$ is continuous, then $f \circ r$ necessarily has a fixed point, say $a^*$, which must be in $A$, so that $a^* = f(r(a^*)) = f(a^*)$ is also a fixed point of $f$. ■

C. Here are two basic observations that are too obvious to prove.

Lemma: If $r : X \to A$ and $s : A \to B$ are retractions, then $s \circ r : X \to B$ is a retraction, so $B$ is a retract of $X$.

Lemma: If $A$ is a retract of $X$ and $A \subset Y \subset X$, then $A$ is a retract of $Y$.

D. We say that $A$ is a neighborhood retract in $X$ if $A$ is a retract of an open $U \subset X$.

Lemma: If $A$ is a neighborhood retract in $X$ and $B$ is a neighborhood retract in $A$, then $B$ is a neighborhood retract in $X$.

Proof: Let $r : U \to A$ and $s : V \to B$ be retractions, where $U$ is a neighborhood of $A$ and $V \subset A$ is a neighborhood of $B$ in the relative topology of $A$. The definition of the relative
topology implies that there is a neighborhood $W \subset X$ of $B$ such that $V = A \cap W$. Then $U \cap W$ is a neighborhood of $B$ in $X$, and the composition of $s$ with the restriction of $r$ to $U \cap W$ is a retraction onto $B$. ■

E. A metric space $A$ is an absolute neighborhood retract (ANR) if, whenever $X$ is a metric space and $A' \subset X$ is homeomorphic to $A$, $A'$ is a neighborhood retract in $X$.

1. This sounds like a remarkably strong property, but it isn’t: it turns out that a metric space $A$ is an ANR if it (or its homeomorphic image) is a retract of an open subset $U$ of a convex subset $C$ of a locally convex linear space.

F. A metric space $A$ is an absolute retract (ANR) if, whenever $X$ is a metric space and $A' \subset X$ is homeomorphic to $A$, $A'$ is a retract of $X$.

1. It turns out that an ANR is an AR if and only if it is contractible.

2. Kinoshita gave an example of a nonempty compact contractible set in $\mathbb{R}^3$ that is not an AR and does not have the fixed point property. Thus AR’s represent an outer limit of topological generality for fixed point theory.

III. The Index Axioms

A. The fixed point index is a function that assigns an integer to a correspondence of a certain type, or, perhaps more intuitively, to the set of fixed points of the correspondence.

1. If $X$ is a set, $S \subset X$, and $F : S \to X$ is a correspondence, we denote the set of fixed points of $F$ by

$$\mathcal{FP}(F) := \{ x \in S : x \in F(x) \}.$$ 

2. To avoid certain ambiguities we restrict attention to correspondences that have no fixed points in the boundary of their domains.

**Definition:** An index admissible correspondence for a compact metric space $X$ is an upper semicontinuous correspondence $F : \overline{U} \to X$ whose domain is the closure of an open $U \subset X$
and which has no fixed points on the boundary of $U$:

$$\mathcal{FP}(F) \cap (\overline{U} - U) = \emptyset.$$  

**Definition:** An index base for $X$ is a set $\mathcal{I}$ of index admissible correspondences for $X$ such that $F|_{U'} \in \mathcal{I}$ whenever $F : \overline{U} \to X$ is in $\mathcal{I}$, $U' \subset U$ is open, and $F|_{U'}$ is index admissible.

3. In stating invariance under homotopy, we need to specify the relevant homotopies, which are those that don’t let fixed points pass over the boundary $\overline{U} \setminus U$.

**Definition:** If $U \subset X$ is open, an index admissible homotopy for $X$ and $U$ is a continuous function $t \mapsto h_t \in C(\overline{U}, X)$ such that each $h_t$ is index admissible.

B. We can now state the first batch of axioms.

**Definition:** Let $\mathcal{I}$ be an index base for a compact metric space $X$. An index for $\mathcal{I}$ is a function $\Lambda_X : \mathcal{I} \to \mathbb{Z}$ satisfying:

(I1) (Normalization) If $c : X \to X$ is a constant function, and an element of $\mathcal{I}$, then

$$\Lambda_X(c) = 1.$$  

(I2) (Additivity) If $F : \overline{U} \to X$ is in $\mathcal{I}$, $U_1, \ldots, U_r$ are disjoint open subsets of $U$, and $F$ has no fixed points in $\overline{U} \setminus (U_1 \cup \ldots \cup U_r)$, then

$$\Lambda_X(F) = \sum_i \Lambda_X(F|_{U_i}).$$  

(I3) (Homotopy) $\Lambda_X(h_0) = \Lambda_X(h_1)$ whenever $h : [0, 1] \to C(\overline{U}, X)$ is an index admissible homotopy with $h_0, h_1 \in \mathcal{I}$.

(I4) (Continuity) Each $F : \overline{U} \to X$ in $\mathcal{I}$ has a neighborhood $A$ in the space of upper semicontinuous correspondences $F' : \overline{U} \to X$ (endowed with the upper topology) such that $\Lambda_X(F') = \Lambda_X(F)$ for all $F' \in \mathcal{I} \cap A$. 

4
Remark: Continuity is very closely related to Homotopy, so much so that you should think of these two axioms as different manifestations of a single condition. In this sense our axiom system is redundant. Our method of establishing the index at a very general level is to start with small and very well-behaved index bases on “nice” spaces, then pass to several increasingly general frameworks. It facilitates the argument to have both Homotopy and Continuity available. On the other hand, as we pass to higher levels of generality it becomes not so easy to prove that, for instance, Homotopy is implied by Normalization, Additivity, and Continuity.

D. There are two additional properties of the index that we treat as axioms for similar reasons.

1. The first, Multiplication, pertains to cartesian products.

2. The second, Commutativity, relates the index of the compositions $g \circ f$ and $f \circ g$ when the domain of $g$ is the range of $f$ and the range of $g$ is the domain of $f$.

   a. Unfortunately, the Commutativity property pertains to a circumstance that has a rather cumbersome description.

Definition: A commutation configuration is a tuple $(X, U, V, f, X', U', V', f')$ where $X$ and $X'$ are compact metric spaces and:

(a) $V \subset U \subset X$ and $V' \subset U' \subset X'$ with $U, U', V,$ and $V'$ open;
(b) $f \in C(U, X')$ and $f' \in C(U', X)$ with $f(V) \subset U'$ and $f'(V') \subset U$;
(c) $f' \circ f|_{V}$ and $f \circ f'|_{V'}$ are index admissible;
(d) $f(FP(f' \circ f|_{V})) = FP(f \circ f'|_{V'})$.

3. Multiplication and Commutativity relate indices of functions and correspondences defined on different spaces, so we need to extend our framework.
Definition: An *index scope* $S$ consists of the following data:

(a) a class of compact metric spaces $S_S$ that contains the cartesian product $X \times Y$ whenever $X, Y \in S_S$;
(b) an index base $I_S(X)$ for each $X \in S_S$ such that $F \times G \in I_S(X \times Y)$ whenever $X, Y \in S_S$, $F \in I_S(X)$, and $G \in I_S(Y)$.

E. The complete description of the index is obtained by adding two more axioms.

Definition: An *index* for an index scope $S$ is a specification of an index $\Lambda_X$ for each $X \in S_S$ such that the following conditions are satisfied:

(I5) (Multiplication) If $X, Y \in S_S$, $F \in I_S(X)$, and $G \in I_S(Y)$, then

$$\Lambda_{X \times Y}(F \times G) = \Lambda_X(F) \cdot \Lambda_Y(G).$$

(I6) (Commutativity) If $(X, U, V, f, X', U', V', f')$ is a commutation configuration with $X, X' \in S_S$, $f' \circ f|_{V} \in I_S(X)$, and $f \circ f'|_{V'} \in I_S(X')$, then

$$\Lambda_X(f' \circ f|_{V}) = \Lambda_Y(f \circ f'|_{V'}).$$

F. The main result.

1. Let $S_{S_{\text{Ctr}}}$ be the class of compact absolute neighborhood retracts.
2. For each $X \in S_{S_{\text{Ctr}}}$, let $I_{S_{\text{Ctr}}}(X)$ be the union over open $U \subset X$ of the sets of index admissible upper semicontinuous contractible valued correspondences $F : \overline{U} \to X$.

   a. Since cartesian products of contractible valued correspondences are contractible valued, we have defined an index scope $S_{\text{Ctr}}$.

The Index Theorem: There is a unique index $\Lambda^{\text{Ctr}}$ for $S_{\text{Ctr}}$. 

6
IV. Some Consequences

A. Perhaps the most basic consequence of Additivity is that if $\emptyset_X : \emptyset \to X$ is the correspondence whose domain is the null set, then

$$\Lambda_X(\emptyset_X) = \Lambda_X(\emptyset_X) + \Lambda_X(\emptyset_X) = 0.$$  

1. Applying Additivity again, it follows that $\Lambda_X(F) = \Lambda_X(F|_\emptyset) = 0$ whenever $F : \mathcal{U} \to X$ is an element of $\mathcal{I}$ with $\mathcal{FP}(F) = \emptyset$.

2. That is, the index is about fixed points.

3. If $X$ is a topological space, $Y$ is a set, and $x_0 \in X$, a `germ` of functions from $X$ to $Y$ at $x_0$ is an equivalence class of functions $f : X \to Y$, where the equivalence relation is “agrees with on some neighborhood of $x_0$.”
   a. Generalizing in the obvious way, one can define a “germ” of correspondences at a compact $K \subset X$.
   b. Additivity says that the index is essentially a function of the germs of $F$ at the components of $\mathcal{FP}(F)$.
   c. We may speak of the index of a component $C \subset \mathcal{FP}(F)$ that is isolated (that is, has a neighborhood containing no other fixed points).
   d. While this number depends on more than $F|_C$, it is be determined by the behavior of $F$ on arbitrarily small neighborhoods of $C$.

B. Let $X$ be a compact AR.

1. Since $X$ is contractible, any continuous function is homotopic to a constant function.

2. Applying Homotopy and Normalization, for any continuous function $f : X \to X$ we have $\Lambda_X(f) = 1$ and thus $\mathcal{FP}(f) \neq \emptyset$.

3. That is, a nonempty compact AR has the fixed point property.
C. Suppose that (a) $FP(F)$ has finitely many connected components, and (b) the index of each of them is one.

1. Then there must be exactly one component, i.e., $FP(F)$ is connected.
2. In economics this fact is rarely used to prove equilibrium uniqueness results because almost always there are more direct methods.
3. Eraslan and McLennan (2004) is an example of a uniqueness result for which no other method of proof is currently known.

D. Clearly Continuity implies that the index of an inessential set of fixed points is zero.

1. In well behaved settings the converse holds: if a connected component of the set of fixed points is isolated and its index is zero, then it is inessential.

E. Suppose $F \in U \to X$ is in $I$, and $FP(F) = K_1 \cup \cdots \cup K_r$ where $K_1, \ldots, K_r$ are compact, pairwise disjoint, and connected.

1. Then there are pairwise disjoint open sets $U_1, \ldots, U_r \subset U$ with $K_j \subset U_j$ for each $j$.
   a. Since $X$ is a metric space, it is an easy exercise to find suitable $U_1, \ldots, U_r$.
2. Additivity says that
   \[
   \Lambda_X(F) = \Lambda_X(F|_{U_1}) + \cdots + \Lambda_X(F|_{U_r}),
   \]
   so the index of $F$ is the sum of the indices of the components $K_1, \ldots, K_r$, or, more precisely, of the restrictions of $F$ to arbitrarily small (since $U_j$ could be replaced by any smaller neighborhood of $K_j$) neighborhoods of $K_1, \ldots, K_r$.
3. In particular, if $\Lambda_X(F) \neq 0$, then at least one of $K_1, \ldots, K_r$ is essential.

F. Let $N = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ be a normal form game.

1. If $u$ lies in a generic set, then there are finitely many equilibria, all of which are regular.
2. It turns out that the index of a regular equilibrium must be either 1 or $-1$.
3. For generic $u$ all equilibria are strict: playing a pure strategy that is assigned probability zero yields strictly less than the equilibrium expected utility.
4. The restriction of the best response correspondence to a neighborhood of a strict pure equilibrium is a constant function.
   a. Applying Additivity and Normalization, the index of each pure equilibrium is 1.

5. Arguing along these lines, Gul, Pearce, and Stacchetti showed that for generic \( u \) the number of mixed equilibria is not less than the number of pure equilibria minus one.

**Research Problem:** In one of my papers I show that if \( \#S_1 \geq \max\{\#S_2, \ldots, \#S_n\} \), then the maximal number of strict pure equilibria is \( \#S_2 \times \cdots \times \#S_n \). For games with this many strict pure equilibria, what is the minimal number of mixed equilibria?

G. Recently Demichelis and Ritzberger showed that an equilibrium of index \(-1\) cannot be stable under any “reasonable” dynamic adjustment process.

H. The degree of a set of equilibria of a normal form game \( N = (S_1, \ldots, S_n; u_1, \ldots, u_n) \) is a concept defined in terms of the equilibrium correspondence.

1. Govindan and Wilson, and also Demichelis and Ritsberger, have shown that the degree and the index of a component of the set of equilibria agree.

V. How the Proof Works

A. The first step is to establish index theory in the best possible setting: \( M \) is a smooth manifold with boundary, \( U \subset M \) is open, and \( f : U \to M \setminus \partial M \) is a smooth index admissible function whose fixed points are all regular.

1. The hard part of the argument is to establish Homotopy. The argument involves some important concepts of differential topology.
B. The other steps are “bootstraps”: using the fact that index theory is valid at one level of generality, we extend it to a still more general setting. There are two main ideas:

1. After one shows that a large class of spaces can be approximated, in a certain sense, by simple spaces, Commutativity is used to extend index theory from the simple spaces to the complex ones. (This idea appeared in the Ph.D. thesis of Felix Browder, who is a former president of the American Mathematical Association.)

2. After one shows that well behaved functions are dense (in the upper topology) in the space of not-so-well behaved correspondences, Continuity is used to extend index theory from the well behaved functions to the correspondences.