I. Introduction.

A. We now begin the theory of games in extensive form.

1. This is more concrete. Most economic models are described in this way. The extensive form is also closer to the "rules" of a game as the term would usually be understood.

2. Notationally these games are much more complex. We must have symbols for the following types of objects.
   a. States of the game.
   b. Information sets.
   c. Actions, and the set of actions available at each information set.
   d. The assignment of an agent to each information set.
   e. The rule assigning a new state when an allowed action is chosen at a given state.
   f. The payoffs at game-ending states.

B. The transition from normal form games to extensive games is eased if we begin by considering games of perfect information.

1. Examples include chess, checkers, go, and backgammon.

2. The key feature is that the agent choosing the move always knows the state of the game.

C. In addition to being rather simple, this class of games is very interesting in its own right.

1. Historically the first work on extensive games focused on this case.
2. Equilibrium theory for these games is well understood and serves as a useful benchmark for evaluating solution concepts for general extensive games.

II. The Tree.

A. The notation developed here will also be used in the study of games of incomplete information.

B. Trees can be defined in several ways – we give two.

C. The first definition

1. Let $T$ be a finite set of nodes, and let $\prec$ be a strict partial ordering of $T$ denoting precedence. For $t \in T$ let

   $$P(t) = \{x \in T | x \prec t\}$$

   be the set of predecessors of $t$.

2. **Definition:** We say that the pair $(T, \prec)$ is an arborescence if, for all $t \in T$, $P(t)$ is completely ordered by $\prec$, i.e.

   $$x_1, x_2 \in P(t) \Rightarrow [x_1 \prec x_2 \text{ or } x_1 = x_2 \text{ or } x_2 \prec x_1].$$

3. If $(T, \prec)$ is an arborescence, let

   $$W = \{w \in T | P(w) = \emptyset\}$$

   be the set of initial nodes.

**Lemma:** $W$ is nonempty.

**Proof:** Otherwise $t \prec t$ for some $t$, an impossibility. ■

4. **Definition 2:** A tree is an arborescence with exactly one initial node $w$.

D. The immediate predecessor function.

1. For $t \in T - \{w\}$ let $p_1(t) = \max P(t)$ be the immediate predecessor of $t$.

   Note that
2. There is usually no need to do so, but it is sometimes convenient to set $p_1(w) = w$. With this convention we can define $p_\ell : T \to T (\ell = 2, 3, \ldots)$ by $p_\ell(t) = p_1(p_{\ell-1}(t))$. Then for all $t \in T$ there exists $\ell$ such that $p_\ell(t) = w$.

3. Conversely, suppose there is a function $p : T \to T$ with $p(w) = w$ such that for all $t \in T$ there is a natural number $\ell$ with $p_\ell(t) = w$. Then $(T, \prec)$ is a tree if we define the precedence relation $\prec$ by declaring that $x \prec t$ if and only if $x \neq t$ and there is an $\ell$ such that $x = p_\ell(t)$. This is our second definition.

E. Other derived notation.

1. $F(t) = p_1^{-1}(t) = \{t' | p_1(t') = t\}$ is the set of immediate successors of $t$.

2. $Z = \{z \in T | F(z) = \emptyset\}$ is the set of terminal nodes.

3. $X = T - Z$ is the set of strategic nodes.

4. $Y = T - W$ is the set of noninitial nodes.

5. $Z(x) = \{z \in Z | x \prec z\}$ is the set of terminal successors of $x$.

III. Games of Perfect Information.

A. Definition: A game of perfect information is a tuple

$$G = ((T, \prec), (I, \iota), u)$$

where

1. $(T, \prec)$ is a tree,

2. $I$ is a finite set of agents,

3. $\iota : X \to I$ is a function, and

4. $u = (u_i)_{i \in I}$ is a vector of utility function $u_i : Z \to \mathbb{R}$. (Alternatively, we may regard $u$ as an element of $\mathbb{R}^{Z \times I}$.)

B. Informally, the rules of $G$ are that $\iota(w)$ chooses $t_1 \in F(w)$, $\iota(t_1)$ chooses $t_2 \in F(t_1)$, and so on until a terminal node is reached.
**IV. Derived Normal Forms.**

A. Let $X_i = \iota^{-1}(i) = \{x \in X | \iota(x) = i\}$.

B. **Definition:** A *pure strategy* for $i$ is a function $s_i : X_i \rightarrow T$ with $s_i(x) \in F(x)$ for all $x \in X_i$. Let $S_i$ be the set of pure strategies for $i$, and let $S = \Pi_{i \in I} S_i$.

**Very Important Remark:** A strategy vector $s$ may be regarded as a function $s : X \rightarrow T$ satisfying $s(x) \in F(x)$ for all $x \in X$. Conversely, any such function may be regarded as a strategy vector.

C. **Definition:** If $s$ is a strategy vector, define $t_0(s) = w, t_1(s) = s(w), t_2(s) = s(t_1(s)), \text{ and so on.}$ The *terminal node determined by* $s$ is $z(s) = t_{\ell(s)}(s)$ where $\ell(s)$ is the integer with $t_{\ell(s)}(s) \in \mathbb{Z}$.

D. **Definition:** The *normal form* of $G$ is

$$N(G) = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$$

where the strategy sets are those defined above and, abusing notation, $u_i(s) = u_i(z(s))$.

E. **Definition:** The *agent normal form* of $G$ is

$$AN(G) = (X, (F(x))_{x \in X}, (\bar{u}_x)_{x \in X})$$

where $\bar{u}_x(s) = u_{i(x)}(z(s))$. (Remark: A strategy vector for the agent normal form is a function $s : X \rightarrow T$ with $s(x) \in F(x)$ for all $x$, so, with the obvious identifications, the set of pure strategy vectors is the same in $N(G)$ and $AN(G)$.)

F. **Definition:** Two strategies $s_i, s'_i \in S_i$ are *realization equivalent* if $z(s_i, s_{-i}) = z(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

1. Realization equivalence is an equivalence relation.
2. Let $S_i$ be the set of equivalence classes for agent $i$.
3. If $s = (s_i)_{i \in I} \in S = \Pi_{i \in I} S_i$, we can unambiguously define $z(s)$ by $z(s) = z(s)$ where $s$ is any strategy vector with $s_i \in S_i$ for all $i$. 

4
4. Abusing notation again, let \( u_i(s) = u_i(z(s)) \).

G. Definition: The reduced normal form of \( G \) is
\[
RN(G) = (I, (S_i)_{i \in I}, (u_i)_{i \in I}).
\]

H. An example.
1. Two agents take turns choosing between “terminate” of “continue” until “terminate” is chosen or until “continue” has been chosen 20 times.
2. In \( N(G) \) there are \( 2^{20} \approx 10^6 \) strategy vectors.
3. In \( AN(G) \) there are the same \( 2^{20} \approx 10^6 \) strategy vectors.
4. In \( RN(G) \) there are \( 11^2 = 121 \) strategy vectors.

V. Subgames and Subgame Perfect Equilibrium

A. Introduction
1. Common knowledge of rationality implies that rationality should be expected in all contingencies.
2. This motivates interest in subgames, and notions of equilibrium based on them.

B. For \( x \in X \) let
\[
T^x = \{ x \} \cup \{ T|x \rightarrow t \},
\]
\[
\prec^x = \prec \cap (T^x \times T^x),
\]
\[
i^x = i|_{X \cap T^x},
\]
\[
u^x_i = u_{i|_{T^x \cap Z}}.
\]

C. Definition: The subgame beginning at \( x \) is
\[
G^x = ((T^x, \prec^x), (I, i^x), u^x)
\]
1. If \( s_i \in S_i \) is a normal form strategy for \( i \) in \( G \), let \( s^x_i = s_i|_{X_i \cap T^x} \).
2. If \( s \in S \) let \( s^x = (s^x_i)_{i \in I} \).

D. Definition: A pure strategy subgame perfect equilibrium for \( G \) is a normal form strategy vector \( s \) such that, for all \( x \), \( s^x \) is a Nash equilibrium of \( N(G^x) \).

Theorem: Every game of perfect information has at least one pure strategy subgame
perfect equilibrium.

**Proof:** Let \( x \) be a strategic node such that \( G^{x'} \) has a p.s.s.p.e. for all \( x' \in X \) with \( x \prec x' \). Let \( F(x) = \{t_1, \ldots, t_J\} \), and for those \( t_j \) in \( X \) let \( s_j^t \) be a p.s.s.p.e. of \( G_j^t \). Define \( z(t_j) \) to be \( t_j \) if \( t_j \in Z \), and otherwise let \( z(t_j) = z(s_j^t) \). Choose
\[
s_i(x) \in \text{argmax}_{t \in F(x)} u_i(z(t_j)).
\]
We have defined all components of a strategy vector \( s : T^x \cap X \to T^x \). It is obvious that \( s \) is a Nash equilibrium of \( G^x \), so \( s \) is a p.s.s.p.e.

**E. Definition:** A game \( G \) is **without indifference between outcomes** if there is a space of outcomes \( \Omega \) and maps \( \omega : Z \to \Omega \) and \( v_i : \Omega \to \mathbb{R} \) such that \( u_i = v_i \circ \omega \) and \( v_i(\omega) \neq v_i(\omega') \) whenever \( \omega \neq \omega' \).

1. In chess, for example, there are many terminal nodes, but \( \Omega = \{\text{win, loss, draw}\} \).

**Proposition 1:** (Zermelo-Kuhn) If \( G \) is without indifference between outcomes, for all \( x \in X \) there is \( \omega^x \in \Omega \) such that \( \omega(z(s^x)) = \omega^x \) for all p.s.s.p.e. \( s^x \) of \( G^x \).

**Proof:** As in the proof above, let \( F(x) = \{t_1, \ldots, t_J\} \) and suppose that each \( t_j \) is either terminal, in which case we set \( \omega_j^t = \omega(t_j) \), or has \( \omega_j^t \in \Omega \) with \( \omega(z(s_j^t)) = \omega_j^t \) for all p.s.s.p.e. \( s_j^t \). If \( s^x \) is a p.s.s.p.e. of \( G^x \) then
\[
s^x(x) = \text{argmax}_{t \in F(x)} v_i(z(t_j)) \text{ and } \omega(z(s^x)) = \omega^x = \text{argmax}_{t \in F(x)} v_i(\omega^t).
\]
The claim now follows by backward induction.

**F. Definition:** A game \( G \) is **without indifference between terminal nodes** if \( u_i(z) \neq u_i(z') \) whenever \( z \neq z' \).

**Proposition 2:** If \( G \) is without indifference between terminal nodes, then there is a unique p.s.s.p.e.
Proof: Let $\Omega = Z$ in Proposition 1. $\blacksquare$