Rationalizable Strategies

I. Introduction
   A. Dominance is a very strong notion.
   B. A somewhat weaker notion is mixture domination: a pure strategy is mixture dominated if it has a lower expected payoff than some mixed strategy regardless of what other agents do.
   C. Even weaker is the notion of a strategy that is not rationalizable in the sense that there is no vector of mixed strategies for the other agents that makes the strategy an optimal choice. We will explore the consequences of iterative elimination of strategies according to this principle.

II. Philosophy
   A. In the traditional (since von Neumann and Morgenstern) interpretation of a noncooperative game, it is a situation that occurs only once and represents all interactions between the agents involved.
      1. In particular, a game is not to be confused with an infinitely repeated version of itself.
      2. Rewards in later interaction and “reputation building” are ruled out in this interpretation.
   B. In this interpretation it is hard to see how the agents’ mixed strategies could be common knowledge, as is assumed in the Nash equilibrium concept, as we will see.
      1. There is another interpretation of a game in which it is a social situation that occurs repeatedly in society, but always between agents who have not met before and who do not expect to encounter each other again.
a. Reputation building and retaliation are impossible.

b. Since agents see the history of the game in society, they can form expectations about the other agents’ behavior.

C. Bernheim and Pearce explore the consequences of the assumption that the rationality of all agents is common knowledge without also assuming that the agents’ mixed strategies are common knowledge. This seems appropriate for the one shot interpretation.

III. Definitional Preliminaries

A. For any finite set $X$ let $\Delta(X) = \{\mu : X \to [0, 1] | \sum_{x \in X} \mu(x) = 1\}$.

B. Let $N = \langle S_1, \ldots, S_n; u_1, \ldots, u_n \rangle$ be a game in normal form. The set of mixed strategies for agent $i$ is $\Delta(S_i)$. The set of mixed strategy vectors is $\Sigma = \prod_{i \in I} \Delta(S_i)$.

Let $\Sigma_i = \Delta(S_1) \times \cdots \times \Delta(S_i-1) \times \Delta(S_{i+1}) \times \cdots \times \Delta(S_n)$, and for $\sigma_i \in \Delta(S_i)$ and $\sigma_{-i} \in \Sigma_{-i}$, let $\langle \sigma_i, \sigma_{-i} \rangle \in \Sigma$ be the obvious vector of mixed strategies.

C. The expected payoff for agent $i$ at a mixed strategy vector $\sigma$ is

$$u_i(\sigma) = \sum_{s \in S} (\prod_{j \in I_i} \sigma_j(s_j)) \cdot u_i(s)$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \cdot [\sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i})]$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}).$$

1. Remark: The function $u_i : \Sigma \to \mathbb{R}$ is obviously continuous.

D. We say that $\sigma'_i$ is a best response for $i$ to $\sigma \in \Sigma$ (actually to $\sigma_{-i} \in \Sigma_{-i}$) if $u_i(\sigma'_i, \sigma_{-i}) \geq u_i(\sigma'', \sigma_{-i})$ for all $\sigma'' \in \Delta(S_i)$.

1. Let $BR_i(\sigma)$ be the set of best responses for $i$ to $\sigma$.

2. Let $BR(\sigma) = \prod_{i \in I} BR_i(\sigma) \subset \Sigma$. The set valued mapping $BR : \Sigma \to \Sigma$ is called the best response correspondence.

E. Lemma: $\sigma'_i \in BR_i(\sigma)$ if and only if $/sigma'\$ assigns positive probability only to pure strategies in $BR_i(\sigma)$. 

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\textbf{Proof:} As we saw above, \( u_i(\sigma) \) is linear in \( \sigma_i \). ■

\section*{IV. Mixture Domination}

\textbf{A. Definition:} \( s_i \in S_i \) is strongly mixture dominated by \( \sigma_i \in \Delta(S_i) \) if
\[
u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}) \quad \text{for all} \quad \sigma_{-i} \in \Sigma_{-i} \quad \text{or, equivalently,} \nu_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all} \quad s_{-i} \in S_{-i}.
\]

\textbf{B.} Following the ideas presented in the last lecture, it is possible to define a sequence of strong mixture domination truncations. We say that such a sequence is complete if no further strong mixture domination truncations are possible. The argument used last time shows that all complete sequences of strong mixture domination truncations have the same final game.

\textbf{C.} Similarly, the notion of weak mixture domination can be defined in the obvious way. Since pure weak domination is a special case, the example discussed last time shows that iterative elimination of weakly mixture dominated strategies is not well defined.

\section*{V. Rationalizability}

\textbf{A. Definition:} If \( H_i \subseteq \Delta(S_i) \) is given for \( i = 1, \ldots, n \), we define \( H_i(t) \) inductively by \( H_i(0) = H_i \) and, for \( t = 1, 2, \ldots \),
\[
H_i(t) = \{ \sigma_i \in H_i(t-1) | \sigma_i \in BR_i(\tau) \text{ for some } \tau \in \Pi_{j \neq i} \text{Con}(H_j(t-1)) \}.
\]

1. \textbf{Notation:} For \( X \subseteq IR^m \), \( \text{Con}(X) \) denotes the convex hull of \( X \).

2. Given \( H_1, \ldots, H_n \) let \( R_i(H_1, \ldots, H_n) = \bigcap_{t=0,1,2,\ldots} H_i(t) \).

\textbf{B. Definition:} The set of rationalizable strategies for player \( i \) is
\[
R_i(\Delta(S_1), \ldots, \Delta(S_n))
\]

\textbf{C. Definition:} A set \( A_i \subseteq \Delta(S_i) \) has the pure strategy property if \( s_i \in A_i \) whenever there is \( \sigma_i \in A_i \) with \( \sigma_i(s_i) > 0 \).
**Proposition 1:** Suppose that, for all $i$, $H_i$ is compact and has the pure strategy property. Then each $H_i(t)$ is compact and has the pure strategy property. Moreover, there is $k$ such that $H_i(t) = H_i(k)$ for all $t \geq k$.

**Proof:** Let $\{\sigma_i^r\}_{r=1,2,\ldots}$ be a sequence in $H_i(t)$ converging to $\sigma_i$. By induction we may assume that $H_i(t-1)$ is closed, so $\sigma_i \in H_i(t-1)$. For each $i$ there is $\tau^r \in \prod_{j \in I} \text{Con}(H_j(t-1))$ such that $\sigma_i^r \in \text{BR}_i(\tau^r)$. Taking a subsequence if necessary, we may assume that $\tau^r \to \tau$ and $\tau \in \prod_{j \in I} \text{Con}(H_j(t-1))$. If $\sigma_i$ is not an element of $\text{BR}(\tau)$ then $u_i(\sigma_i, \tau_{-i}) < u_i(\sigma_i^r, \tau_{-i})$ for some $\sigma_i^r \in \Delta(S_i)$, but since $u_i : S \to \mathbb{R}$ is continuous, this would imply that $u_i(\sigma_i^r, \tau_{-i}) < u_i(\sigma_i^r, \tau_{-i}^r)$ for large $r$, contradicting $\sigma_i^r \in \text{BR}(\tau^r)$. Thus $\sigma_i \in H_i(t)$, and this shows that $H_i(t)$ is closed and thus compact.

Assume that $H_i(t-1)$ has the pure strategy property. If $\sigma_i \in H_i(t)$, then $\sigma_i \in H_i(t-1)$, so all pure strategies $s_i$ with $\sigma_i(s_i) > 0$ are in $H_i(t-1)$. But they must also all be best responses to any $\tau \in \prod_{j \in I} \text{Con}(H_j(t-1))$ such that $\sigma_i \in \text{BR}_i(\tau)$, so they are all in $H_i(t)$. Thus $H_i(t)$ has the pure strategy property.

From the definition we see that $H_i(t)$ cannot be different from $H_i(t-1)$ unless $\prod_{j \in I} \text{Con}(H_j(t-1))$ is different from $\prod_{j \in I} \text{Con}(H_j(t-2))$. By virtue of the pure strategy property, each set $\prod_{j \in I} \text{Con}(H_j(t))$ is a cartesian product of faces of the simplices of mixed strategies. Since there are only finitely many such faces, this establishes the last claim, namely that eliminations cease after finitely many stages. ■

**Proposition 2:** For all $i$, $R_i(\Delta(S_1), \ldots, \Delta(S_n)) \neq \emptyset$.

**Proof:** Let $H_i = \Delta(S_i)$. An easy induction argument establishes that, for each $i$ and $t$,

$$H_i(t) = \text{BR}_i(\prod_{j \in I} \text{Con}(H_j(t-1))).$$

That each $H_i(t)$ is nonempty now follows from induction, and the claim follows from the result above that elimination ceases after finitely many stages, or, if you prefer, from the
fact that a descending sequence of nonempty compact sets has a nonempty intersection. ■

D. Definition: A collection of nonempty sets \( A_i \subset \Delta(S_i), \ i \in I, \) has the best response property if, for all \( i \) and \( \sigma_i \in A_i, \sigma_i \in BR_i(\tau) \) for some \( \tau \in \Pi_{j \in I} \text{Con}(A_j) \).

**Proposition 3:** The collection of sets \( R_i(\Delta(S_1), \ldots, \Delta(S_n)), \ i \in I, \) has the best response property.

**Proof:** This is a consequence of the termination of the truncation process after finitely many stages. ■

We now wish to show that the sets \( R_i(\Delta(S_1), \ldots, \Delta(S_n)) \) do not really depend on the particular order of truncation.

**Proposition 4:** If \( \{A_i\}_{i \in I} \) is a collection with the best response property, and \( A_i \subset H_i \) for all \( i \), then \( A_i \subset H_i(t) \) for all \( i \) and \( t \).

**Proof:** By induction it suffices to show that \( A_i \subset H_i(1) \), and this is an immediate consequence of our definitions. ■

We summarize our results in the following Theorem, which is an immediate consequence of what we have already proved.

**Theorem:** Any collection \( \{X_i\}_{i \in I} \) with the best response property has \( X_i \subset R_i(\Delta(S_1), \ldots, \Delta(S_n)) \) for all \( i \).

**VI. Comparison with Strict Mixture Domination**

Clearly iterative elimination of unrationalizable strategies is at least as powerful a procedure as iterative elimination of strictly mixture dominated strategies. We now present an example due to Pearce that shows that the containment is strict. There are three agents
with strategy sets \( S_1 = \{\ell, m, r\} \), \( S_2 = \{U, D\} \), and \( S_3 = \{H, T\} \). The payoffs for agent 1 are given in the following matrices.

\[
\begin{array}{cccc}
& H & T \\
U & 6 & 6 & 6 & 6 \\
m & 10 & 10 & m & 10 \\
r & 0 & 10 & r & 10 \\
\end{array}
\]

Here \( \ell \) is not dominated, nor is it strictly mixture dominated since any mixture of \( m \) and \( r \) has an expected payoff no greater than 5 in either the case \((H, U)\) or the case \((T, D)\). Nonetheless \( \ell \) is not rationalizable, since if it were rationalizable there would be \( p = \text{Prob}(U) \) and \( q = \text{Prob}(H) \) such that

\[
6 \geq 10 \cdot \max \{1 - pq, 1 - (1 - p)(1 - q)\}.
\]

But this inequality is impossible, since if \( p + q \leq 1 \), then \( 1 - pq > 1 - (1/2)(1/2) = 3/4 \), and if \( p + q \geq 1 \), then \( 1 - (1 - p)(1 - q) \geq 1 - (1/2)(1/2) = 3/4 \).