A Treatise on the Theory of Fixed Points

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September 27, 2004
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Chapter 1

The End of the Story

This treatise presents a self-contained development of the theory of the fixed point index that makes no reference to concepts from algebraic topology. It replaces my earlier manuscript titled Selected Topics in the Theory of Fixed Points. My plan is to fold into a larger work, tentatively title The Mathematics of Nash Equilibrium: I. Fixed Point Theory, which will describe the relationship between fixed point theory and some of the ideas in the stream of research generally described as “refinements of Nash equilibrium.”

The logical prerequisites are quite modest: the reader is expected to have a reasonable background in multivariate calculus and point set topology at the level of advanced undergraduate courses. I have tried to keep such a reader in mind throughout, with the hope that she might obtain brief introductions to several concepts and methods that are important in mathematical economics, and mathematics more generally. In my experience such an approach is complementary to standard mathematical pedagogy in which a single field is developed in depth, with little or no reference to its relationships with the rest of mathematics.

While the material mostly has a straightforward character, there is an “advanced” flavor to much of it, and some of the proofs are unapologetically complicated. The vast majority of the math books I have attempted ended up gathering dust after a few pages or a couple chapters, either because I encountered a roadblock to understanding, or I was distracted by “life” and lost the thread. For such readers I hope that the portion you do manage to absorb will, in the end, seem worthwhile, and that you will manage to maintain a favorable view of the author’s sincerity, even if his expository skills seem doubtful.

Researchers in economics and game theory will mostly wish to use this
1.1. THE FIRST FIXED POINT THEOREMS

as a source and reference for the main result: existence and uniqueness of a fixed point index for contractible valued correspondences. Morally, I share mathematicians’ distrust of the “logical” attitude of many economists, who see nothing wrong with applying theorems whose proofs they have never attempted to understand. But in this case it is at least possible to quickly give a completely rigorous statement of our main results, and this will be a useful introduction to the main concepts for those who read further.

1.1 The First Fixed Point Theorems

Here is the statement of Kakutani’s fixed point theorem:

**Theorem 1.1.1.** If $C \subset \mathbb{R}^n$ is nonempty, compact, and convex, and $F : C \to C$ is an upper semicontinuous convex valued correspondence, then $F$ has a fixed point.

Many readers have seen this before, and know what the various terms here mean. For the rest, here are some brief explanations.

If $X$ and $Y$ are sets, a correspondence $F : X \to Y$ is a function from $X$ to the nonempty subsets of $Y$. Correspondences figure prominently in the mathematical foundations of economics and game theory because the decision problems that are the “atoms” of economic analysis often have multiple solutions. For the most part economic modelling avoids decision problems for which there are no optimal strategies. Accordingly, this definition declares that we have little interest in set valued mappings that allow the empty set as a value.

If $X$ and $Y$ are topological spaces, a correspondence $F : X \to Y$ is upper semicontinuous if:

(a) for each $x \in X$, $F(x)$ is compact;

(b) for each $x_0 \in X$ and each neighborhood $V \subset Y$ of $F(x_0)$ there is a neighborhood $U \subset X$ such that $F(x) \subset V$ for all $x \in U$.

It turns out that if $X$ and $Y$ are metric spaces and $Y$ is compact, then $F$ is upper semicontinuous if and only if its graph

$$\text{Gr}(F) := \{ (x, y) \in X \times Y : y \in F(x) \}$$

is closed. In economics this condition is commonly taken as definition, as in [Deb59]. [CHECK] For the reader who likes to learn by solving problems, proving this is a suitable exercise. A somewhat simpler exercise is to show
that: (a) if \( F \) is upper semi-continuous and each of its values \( F(x) = \{ f(x) \} \) is a singleton, then the associated function \( f : X \to Y \) is continuous; (b) if \( f : X \to Y \) is a continuous function, then the associated correspondence \( F(x) := \{ f(x) \} \) is upper semicontinuous.

If \( Y \subseteq \mathbb{R}^m \) (or, for matter, an infinite dimensional vector space) then \( F \) is **convex valued** if, for all \( x \in X \), \( F(x) \) is convex. Explanations have to end at some level; here the reader will be assumed to know what a convex set is.

### 1.2 “Fixing” Brouwer’s Theorem

Mathematicians strive to craft theorems that maximize the strength of the conclusions while minimizing the strength of the assumptions. One reason for this is obvious: a stronger theorem is a more useful theorem. More important, however, is the desire to achieve a proper understanding of the principle the theorem expresses, and to achieve an expression of this principle that is unencumbered by useless clutter. When a theorem that is “too weak” is proved using methods that “happen to work” there is a strong suspicion that attempts to improve the theorem will uncover important new concepts. This treatise tells a story in which a sequence of concepts are introduced in the hope that they might shed greater light on the problem of “fixing” Kakutani’s theorem. Most of these concepts have vastly greater significance and application than we are able to describe here.

To begin with, let’s think about what’s “wrong” Brouwer’s fixed point theorem, which is the special case of Kakutani’s theorem in which the correspondence \( F \) is derived from a continuous function \( f : C \to C \), so that \( F(x) = \{ f(x) \} \) for each \( x \in C \). A topological space \( X \) **has the fixed point property** if every continuous function \( f : X \to X \) has a fixed point. Brouwer’s theorem gives a sufficient condition (nonempty, compact, convex) for the fixed point property. There seems to be no hope of finding conditions that are both necessary and sufficient for the fixed point property, so the idea of what constitutes an “improvement” of Brouwer’s theorem is, to some extent at least, subjective and aesthetic. Everyone would be happy with a modification that simultaneously weakened the hypotheses while simplifying the proof. Nobody would like a proliferation of special cases. Economists would like the statement of the theorem to be simple, subject to the requirement that it be general enough to encompass the main economic applications. (One might think that economists would be willing to trade generality for simplicity of the proof, but in fact there are very few texts, even at the
1.2. “FIXING” BROUWER’S THEOREM

At the graduate level, that prove Brouwer’s theorem! Mathematicians pride themselves in being eager to absorb any degree of complexity so long as it brings them even a little bit closer to the “essence” of the matter.

With Brouwer’s theorem there is a lingering suspicion that the fixed point property should depend on some property weaker than convexity. The conclusion is topological, so a geometric hypothesis seems out of place. Of course “convex” can be weakened to “homeomorphic to a convex set,” but this is hardly satisfying.

In fact it is pretty easy to generalize this hypothesis in a significant way. If $X$ is a topological space, a subset $A \subset X$ is a retract if there is a continuous function $r : X \to A$ with $r(a) = a$ for all $a \in A$. If $X$ has the fixed point property, then so does $A$, because if $f : A \to A$ is continuous, then so is $f \circ r : X \to A \subset X$, so $f \circ r$ has a fixed point which necessarily lies in $A$ and is consequently a fixed point of $f$. It is easy to construct compact retracts of convex sets that are not themselves homeomorphic to convex sets. One simple example is a disk with a line segment sticking out of it:

$$ \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} \cup \{ (x, 0) : 1 \leq x \leq 2 \}. $$

Let’s introduce a couple more concepts. A homotopy is a continuous function $h : X \times [0,1] \to Y$ where $X$ and $Y$ are topological spaces. It is psychological natural to think of the second variable in the domain as representing time, and we let $h_t := h(\cdot, t) : X \to Y$ denote “the function at time $t$.” We think of $h$ as a process that continuously deforms a function $h_0$ into $h_1$, and we say that two functions $f, g : X \to Y$ are homotopic if there is a homotopy $h$ with $h_0 = f$ and $h_1 = g$. Evidently this is an equivalence relation, and the equivalence classes are called homotopy classes. Another intuitive picture is that $h$ is a continuous path in the space $C(X,Y)$ of continuous function from $X$ to $Y$. As we will see in Chapter ???, this intuition can be made completely precise: [at least when $X$ and $Y$ are well behaved spaces] there is a topology on $C(X,Y)$ such that a continuous path $h : [0,1] \to C(X,Y)$ is the same thing as a homotopy. An important theme of topology is the search for properties of functions that are “invariant under homotopy,” meaning that one endpoint of a homotopy has the property if and only if the other endpoint has it. It can happen that a space $X$ (e.g., the circle $S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$) does not have the fixed point property, but in spite of this every element of some homotopy class of functions from $X$ to itself (e.g., the functions homotopic to the map $(x,y) \mapsto (x,-y)$ on $S^1$) has a fixed point. The celebrated Lefschetz fixed point theorem is one result giving conditions under which this is the case.
A space \( X \) is **contractible** if the identity function \( \text{Id}_X \) on \( X \) is homotopic to a constant function. That is, there is a homotopy \( c : X \times [0,1] \to X \) such that \( c_0 = \text{Id}_X \) and \( c_1(X) \) is a single point. If \( X \) is contractible, a continuous function \( f : X \to X \) can be deformed into a constant function in at least two ways: \( t \mapsto c_t \circ f \) and \( t \mapsto f \circ c_t \) are both homotopies between \( f \) and a constant function. It seems natural to guess that nonempty compact contractible spaces have the fixed point property.

Such a result would be conceptually satisfying because a space homeomorphic to a contractible space is contractible (of course) and a retract of a contractible space is contractible. Specifically, if \( c : X \times [0,1] \to X \) is a contraction and \( r : X \to A \subset X \) is a retraction, then \( t \mapsto r \circ c_t \big|_A \) is a contraction of \( A \). In the case of convexity, the fact that we could obtain a stronger result by prepending the phrases “homeomorphic to a” and “retract of a” suggested that we were dealing with the wrong concept.

But it turns out that a nonempty compact contractible space need not have the fixed point property! In Chapter ??? we will see an example, from [Kin53], of a subset of \( \mathbb{R}^3 \) with these properties. Is there any hope of attaining a result expressing a reasonably general and conceptually satisfying version of the fixed point principle?

An **absolute neighborhood retract** (ANR) is a topological space \( X \) such that whenever \( Y \) is a separable\(^1\) metric space and \( X' \) is a subset of \( Y \) that is homeomorphic to \( X \) (informally, we say that “\( X \) is embedded in \( Y \)”) there is an open neighborhood \( U \subset Y \) of \( X' \) and a retraction \( r : U \to X' \). At first sight this might seem like a remarkably demanding condition, but in fact the class of absolute neighborhood retracts is very large.

An example of a space that is **not** an ANR is the union

\[
K := S^1 \cup \{(t \cos \frac{1}{1-t}, t \sin \frac{1}{1-t}) : 0 \leq t < 1\}
\]

of the unit circle and a curve that spirals outward, approaching the unit circle asymptotically. To see that \( K \) is not an ANR, consider a point \( p \) in the unit circle. For sufficiently small \( \delta > 0 \), the intersection \( K \cap B_\delta(p) \) of \( K \) with the open \( \delta \)-ball around \( p \) is disconnected\(^2\), being an arc from the

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\(^1\)A metric space is **separable** if it has a countable dense subset.

\(^2\)A topological space is **connected** if it does not admit a partition \( \{X_1, X_2\} \) (that is \( X_1 \) and \( X_2 \) are both nonempty, \( X_1 \cap X_2 = \emptyset \) and \( X_1 \cup X_2 = X \)) into sets \( X_1 \) and \( X_2 \) that are both open and closed. More generally, a subset \( S \) of a topological space \( X \) is **connected** if it is connected in its relative topology: there do not exist disjoint open sets \( U_1, U_2 \) such that \( U_1 \cap S \neq \emptyset, U_2 \cap S \neq \emptyset \), and \( S \subset U_1 \cup U_2 \). It is not hard to show that, for any \( x_0 \in X \), the union of all connected sets that contain \( x_0 \) is a connected set. Evidently this set is the maximal connected set containing \( x_0 \). It is called the **connected**
circle together with a countable union of arcs from the curve. If there was
a retraction $r : U \rightarrow A$, where $U$ was an open set containing $K$, then $U$
would contain the open $\varepsilon$-ball $B_\varepsilon(p)$ around $p$ for sufficiently small $\varepsilon$, and
for sufficiently small $\varepsilon$ we would have $B_\varepsilon(p) \cap K \subseteq r(B_\varepsilon(p)) \cap K$ because $r$
is a retraction and $r(B_\varepsilon(p)) \subseteq B_\varepsilon(p) \cap K$ because $r$ is continuous. An easily
proved fact of general topology is that the image of a connected set, under a
continuous function, is connected, so $r(B_\varepsilon(p))$ would have to be connected,
but clearly this is impossible. Roughly, arbitrarily small neighborhoods of $p$
in $K$ contain infinite topological complexity. In contrast an ANR (at least
one embedded in a Euclidean space) must be “locally simple” in the sense
that each point has a neighborhood that is a “smushed down” image of an
open ball.

For us the “right” form of Brouwer’s theorem is: every nonempty compact
contractible ANR has the fixed point property. There is a concept called
acyclicity that is defined in terms of the concepts of algebraic topology. A
contractible set is necessarily acyclic, and the result stated above remains
ture if “contractible” is replaced by “acyclic.” So, in saying that this version
is “right” for us we do not mean that it is the most general version known,
but rather that it represents the proper culmination of the application of
our methods, which are those of point-set topology.

The requirement, in Kakutani’s theorem, that the values of the corres-
donence be convex, is at least as obnoxious as the requirement that the
domain be convex, for pretty much the same reasons. But it can be fixed
more easily. For us the “corrected” version of Kakutani’s theorem is: if
$X$ is a nonempty compact contractible ANR and $F : X \rightarrow X$ is an upper
semicontinuous contractible valued correspondence, then $F$ has a fixed point.
In particular, the values of $F$ do not need to be ANR’s. The main idea of
the argument is to show that any neighborhood of the graph of $F$ contains
the graph of a continuous function $f : X \rightarrow X$, and this happens to be true
even if the values of $F$ are not ANR’s.

1.3 The Index

It might seem like we have already reached a satisfactory and fitting resolu-
tion of “The Fixed Point Problem,” but actually (both in pure mathematics
and in game theory) this is just the beginning. You see, fixed points come
in different flavors.

component of $X$ containing $x_0$. It is easy to see that “in the same connected component
as” is an equivalence relation, so the connected components constitute a partition of $X$. 
[Insert Two Plus One, Inessential, One Minus One figure here.]

The figure above shows the graph of a function \( f : [0, 1] \to [0, 1] \) that has four fixed points. The point \( B \) is a different sort of fixed point from the others, insofar as it can be “perturbed away.” More precisely, for sufficiently small neighborhoods \( U \) of \( B \) it is that the case that, for any neighborhood \( V \subset [0, 1]^2 \) of the graph of \( f \), there is a continuous \( f' : [0, 1] \to [0, 1] \) whose graph is contained in \( V \) and which has no fixed points in \( U \).

There is also a sense in which \( C \) is a different sort of fixed point from \( A \) and \( D \), because \( f \) crosses the diagonal “going from below to above” at \( C \), but crosses it “going from above to below” at \( A \) and \( D \). Roughly, it seems that the number of “downcrossings” of the diagonal should be one more than the number of “upcrossings.”

The main agenda of this treatise is to flesh out these intuitions, making them completely precise, and to express them at a level of generality that represents the outer limits of the power of the methods, from point set topology, that we bring to bear. There is a great deal we could try to say by way of introduction, and necessarily the vast majority will remain unsaid until later. What we will do here is to state the system of axioms characterizing the index, and describe the most general result we will attain.

The fixed point index is a function that assigns an integer to a correspondence of a certain type, or, perhaps more intuitively, to the set of fixed points of the correspondence. If \( X \) is a set, \( S \subset X \), and \( F : S \to X \) is a correspondence, we denote the set of fixed points of \( F \) by

\[
\mathcal{FP}(F) := \{ x \in S : x \in F(x) \}.
\]

To avoid certain ambiguities we restrict attention to correspondences that have no fixed points in the boundary of their domains.

Let \( X \) be a compact metric space.

**Definition 1.3.1.** An index admissible correspondence for \( X \) is an upper semicontinuous correspondence \( F : \overline{U} \to X \) whose domain is the closure of an open \( U \subset X \) and which has no fixed points on the boundary of \( U \):

\[
\mathcal{FP}(F) \cap (\overline{U} - U) = \emptyset.
\]

**Definition 1.3.2.** If \( U \subset X \) is open, an index admissible homotopy for \( X \) and \( U \) is a continuous function \( t \mapsto h_t \in C(\overline{U}, X) \) such that each \( h_t \) is index admissible.

At less general levels the index may be defined explicitly, but our more general characterizations will be axiomatic. The axioms will be in large part...
1.4. FIRST CONSEQUENCES

matters of comparing indices of different sets of fixed points, for different functions and correspondences. In order for the axioms to have sufficient force we need the domain of the index, which is a space of correspondences, to be large enough.

Definition 1.3.3. An index base for $X$ is a set $I$ of index admissible correspondences for $X$ such that $F|_{U'} \in I$ whenever $F : U \to X$ is in $I$, $U' \subset U$ is open, and $F|_{U'}$ is index admissible.

We can now state the first batch of axioms.

Definition 1.3.4. Let $I$ be an index base for a compact metric space $X$. An index for $I$ is a function $\Lambda_X : I \to \mathbb{Z}$ satisfying:

(1) (Normalization\textsuperscript{3}) If $c : X \to X$ is a constant function, and an element of $I$, then

\[ \Lambda_X(c) = 1. \]

(2) (Additivity) If $F : U \to X$ is in $I$, $U_1, \ldots, U_r$ are disjoint open subsets of $U$, and $F$ has no fixed points in $U \setminus (U_1 \cup \ldots \cup U_r)$, then

\[ \Lambda_X(F) = \sum_i \Lambda_X(F|_{U_i}). \]

(3) (Homotopy) $\Lambda_X(h_0) = \Lambda_X(h_1)$ whenever $h : [0, 1] \to C(U, X)$ is an index admissible homotopy with $h_0, h_1 \in I$.

(4) (Continuity) Each $F : U \to X$ in $I$ has a neighborhood $A$ in the (suitably topologized) space of upper semicontinuous correspondences $F' : U \to X$ such that $\Lambda_X(F') = \Lambda_X(F)$ for all $F' \in I \cap A$.

1.4 First Consequences

Let’s think about some of the implications of these axioms. Perhaps the most basic consequence of Additivity is that if $\emptyset_X : \emptyset \to X$ is the correspondence whose domain is the null set, then

\[ \Lambda_X(\emptyset_X) = \Lambda_X(\emptyset_X) + \Lambda_X(\emptyset_X) = 0. \]

\textsuperscript{3}In the literature this condition is sometimes described as “Weak Normalization,” in contrast with a stronger condition defined in terms of concepts from algebraic topology.
Applying Additivity again, it follows that $\Lambda_X(F) = \Lambda_X(F|_\emptyset) = 0$ whenever $F : U \to X$ is an element of $\mathcal{I}$ with $\mathcal{FP}(F) = \emptyset$. That is, the index is about fixed points.

Suppose $F \in U \to X$ is in $\mathcal{I}$ and $\mathcal{FP}(F) = K_1 \cup \cdots \cup K_r$ where $K_1, \ldots, K_r$ are compact, pairwise disjoint, and connected i.e., $K_1, \ldots, K_r$ are the connected components of $\mathcal{FP}(F)$. Then there are pairwise disjoint open sets $U_1, \ldots, U_r \subset U$ with $K_j \subset U_j$ for each $j$. (Since $X$ is a metric space, it is an easy exercise to find suitable $U_1, \ldots, U_r$) Additivity says that

$$\Lambda_X(F) = \Lambda_X(F|_{\overline{U_1}}) + \cdots + \Lambda_X(F|_{\overline{U_r}}),$$

so the index of $F$ is the sum of the indices of the components $K_1, \ldots, K_r$, or, more precisely, of the restrictions of $F$ to arbitrarily small (since $U_j$ could be replaced by any smaller neighborhood of $K_j$) neighborhoods of $K_1, \ldots, K_r$.

We may speak of the index of a component of $\mathcal{FP}(F)$, say $K_j$, and while this number depends on more than $F|_{K_j}$, it is determined by the behavior of $F$ on arbitrarily small neighborhoods of $K_j$.

If every continuous function $f : X \to X$ is homotopic to a constant function in $\mathcal{I}$ (e.g., if $X$ is convex) then Homotopy and Normalization imply that $\Lambda_X(f) = 1$. In this book we will always be dealing with correspondences that can be approximated, in the sense of the topology pertinent to Continuity, by continuous functions. For these reasons it is almost always the case, in settings of interest to economic theory, that $\Lambda_X(F) = 1$ for any $F : X \to X$ in $\mathcal{I}$. Already this implies the fixed point theorem: $\mathcal{FP}(F) \neq \emptyset$ because $\Lambda_X(F) \neq 0$.

But it also gives additional information of potential interest. For instance, if $\mathcal{FP}(F)$ has finitely many connected components, then their indices must sum to one. As one example of how this might be interesting, suppose we could show that $\mathcal{FP}(F)$ has finitely many components and the index of each of them is one. Then there must be exactly one component, i.e., $\mathcal{FP}(F)$ is connected. In economics this fact is rarely used to prove equilibrium uniqueness results because in most settings where equilibrium is unique there are more direct methods, but [EM04] is an example of a uniqueness result for which no other method of proof is currently known.

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4If $X$ is a topological space, $Y$ is a set, and $x_0 \in X$, a germ of functions from $X$ to $Y$ at $x_0$ is an equivalence class of functions $f : X \to Y$, where the equivalence relation is “agrees with on some neighborhood of $x_0.”$ Generalizing in the obvious way, one can define a “germ” of correspondences at a compact $K \subset X$. Additivity says that the index is essentially a function of the germs of $F$ at the components of $\mathcal{FP}(F)$. 
1.5 More Axioms, and the Final Result

Without a precise description of the relevant topology on spaces of correspondences it is hard to say very much about Continuity, but we can indicate that it is very closely related to Homotopy, so much so that you should think of these two axioms as different manifestations of a single condition. In this sense our axiom system is redundant. Our method of establishing the index at a very level is to start with small and very well behaved index bases on “nice” spaces, then pass to several increasingly general frameworks. It facilitates the argument to have both Homotopy and Continuity available. On the other hand, as we pass to higher levels of generality it becomes not so easy to prove that, for instance, Homotopy is implied by Normalization, Additivity, and Continuity.

There are two additional properties of the index that we treat as axioms for similar reasons. The first, “Multiplication,” pertains to cartesian products, while the second, “Commutativity,” relates the index of the compositions $g \circ f$ and $f \circ g$ when the domain of $g$ is the range of $f$ and the range of $g$ is the domain of $f$. These concepts relate indices of functions and correspondences defined on different spaces, so need to extend our framework.

Definition 1.5.1. An index scope $S$ consists of the following data:

(a) a class of compact metric spaces $S_S$ that contains the cartesian product $X \times Y$ whenever $X, Y \in S_S$;

(b) an index base $I_S(X)$ for each $X \in S_S$ such that $F \times G \in I_S(X \times Y)$ whenever $X, Y \in S_S$, $F \in I_S(X)$, and $G \in I_S(Y)$.

Unfortunately, the Commutativity property pertains to a circumstance that has a rather cumbersome description.

Definition 1.5.2. A commutation configuration is a tuple

$$(X, U, V, f, X', U', V', f')$$

where $X$ and $X'$ are compact metric spaces and:

(a) $V \subset U \subset X$ and $V' \subset U' \subset X'$ with $U$, $U'$, $V$, and $V'$ open;

(b) $f \in C(\overline{U}, X')$ and $f' \in C(\overline{U'}, X)$ with $f(\overline{V}) \subset U'$ and $f'(\overline{V'}) \subset U$;

(c) $f' \circ f|_{\overline{V'}}$ and $f \circ f'|_{\overline{V}}$ are index admissible;

(d) $f(F\mathcal{P}(f' \circ f|_{\overline{V'}})) = F\mathcal{P}(f \circ f'|_{\overline{V}})$. 
The complete description of the index concept is obtained by adding two more axioms.

**Definition 1.5.3.** An index for an index scope $S$ is a specification of an index $\Lambda_X$ for each $X \in S_S$ such that the following conditions are satisfied:

(I5) (Multiplication) If $X, Y \in S_S$, $F \in \mathcal{I}_S(X)$, and $G \in \mathcal{I}_S(Y)$, then

$$\Lambda_{X \times Y}(F \times G) = \Lambda_X(F) \cdot \Lambda_Y(G).$$

(I6) (Commutativity) If $(X, U, V, f, X', U', V', f')$ is a commutation configuration with $X, X' \in S_S$, $f' \circ f|_{\overline{U}} \in \mathcal{I}_S(X)$, and $f \circ f'|_{\overline{U'}} \in \mathcal{I}_S(X')$, then

$$\Lambda_X(f' \circ f|_{\overline{U}}) = \Lambda_Y(f \circ f'|_{\overline{U'}}).$$

We can now give the statement of this book’s main result. Let $S_{S^{\text{Ctr}}}$ be the class of compact absolute neighborhood retracts, and for each $X \in S_{S^{\text{Ctr}}}$ let $\mathcal{I}_{S^{\text{Ctr}}}(X)$ be the union over open $U \subset X$ of the sets of index admissible upper semicontinuous contractible valued correspondences in $C(U, X)$. Since cartesian products of contractible valued correspondences are contractible valued, we have defined an index scope $S^{\text{Ctr}}$.

**Theorem 1.5.4.** There is a unique index $\Lambda^{\text{Ctr}}$ for $S^{\text{Ctr}}$. 