Chapter 10

Approximation

Approximation is an idea that lies at the heart of the approach to fixed point theory taken in this book. The upper topology is one important expression of this idea. In this chapter we study approximation of compact ANR’s by simplicial complexes, and we study approximation of contractible valued correspondences \( f : X \to Y \) by functions, where \( X \) and \( Y \) are ANRs.

10.1 Domination

This section’s result shows that a compact ANR can be approximated, with arbitrary accuracy, by simplicial complexes.

**Definition 10.1.1** If \((X,d)\) is a metric space, \(Y\) is a topological space, and \(\varepsilon > 0\), a homotopy \(h : Y \times [0,1] \to X\) is an \(\varepsilon\)-**homotopy** if

\[
d(h(y,s),h(y,t)) < \varepsilon
\]

for all \(y \in Y\) and all \(0 \leq s, t \leq 1\). We say that \(h_0\) and \(h_1\) are \(\varepsilon\)-**homotopic**.

**Definition 10.1.2** For \(\varepsilon > 0\) we say that a topological space \(Y\) \(\varepsilon\)-**dominates** a metric space \((X,d)\) if there are continuous functions \(\varphi : X \to Y\) and \(\psi : Y \to X\) such that \(\psi \circ \varphi : X \to X\) is \(\varepsilon\)-homotopic to \(\text{Id}_X\).

**Theorem 10.1.3** Let \(X\) be a compact ANR. For any \(\varepsilon > 0\) there is a simplicial complex \(K = (V,C)\) such that \(|K|\) \(\varepsilon\)-dominates \(X\).

In preparation for the proof of this we mention the following fact, which shows that this property of compact ANR’s depends only on the topology, not the particular metric.

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Lemma 10.1.4 Let $h : X \to Y$ be a homeomorphism between compact metric spaces $(X,d)$ and $(Y,e)$. For any $\varepsilon > 0$ there is $\delta > 0$ such that $e(h(x), h(x')) < \varepsilon$ whenever $d(x, x') < \delta$.

Proof: Otherwise there are sequences $\{x_j\}, \{x'_j\}$ with $d(x_j, x'_j) < 1/j$ and $e(h(x_j), h(x'_j)) \geq \varepsilon$, and the limit of a convergent subsequence is a point of discontinuity of $h$.

The proof below makes use of the following construction, which is interesting in its own right.

Definition 10.1.5 The nerve of a finite cover $U_1, \ldots, U_n$ of a set $Z$ is the combinatoric simplicial complex $K = (V,C)$ in which $V := \{v_1, \ldots, v_n\}$ and, for each $s \subset V$, $s \in C$ if and only if $\bigcap_{i \in s} U_i \neq \emptyset$.

Proof of Theorem 10.1.2: Lemma ?? above implies that it is enough to show that some metric space homeomorphic to $X$ can be $\varepsilon$-dominated by a simplicial complex. In view of Urysohn’s theorem (Theorem ??) we may assume without loss of generality that $X$ is embedded in the Hilbert cube $I^\infty$. Recall that $I^\infty$ is endowed with the metric, denoted by $d_\infty$, derived from the norm of the Hilbert space $H$. Let $r : U \to X$ be a retraction, where $U \subset I^\infty$ is a neighborhood of $X$.

For $x \in X$ let

$$\eta(x) := \frac{1}{2}d_\infty(x, I^\infty \setminus r^{-1}(B_{\varepsilon/2}(x) \cap X)).$$

Of course $B_{\varepsilon/2}(x) \cap X$ is a neighborhood of $x$ in $X$, and $r$ is continuous, so $r^{-1}(B_{\varepsilon/2}(x) \cap X)$ is a neighborhood of $x$ in $U$. Thus $\eta(x) > 0$. Since $X$ is compact, there is a finite collection of points $x_1, \ldots, x_n$ such that

$$U_1 := B_{\eta(x_1)}(x_1), \ldots, U_n := B_{\eta(x_n)}(x_n)$$

is a cover of $X$. Let $K = (V,C)$ with $V = \{v_1, \ldots, v_n\}$ be the nerve of the cover $U_1, \ldots, U_n$, and let $Y := |K|$.

Let $\nu : X \to (0, \infty)$ be the function

$$\nu(x) := \sum_{i=1}^{n} d_\infty(x, X \setminus U_i).$$
Clearly \( \nu \) is continuous. To realize \(|K|\) as a geometric object we may suppose that \( V = \{v_1, \ldots, v_n\} \), where \( v_1, \ldots, v_n \) are affinely independent points in a Euclidean space. Let \( \varphi : X \to |K| \) be the function

\[
\varphi(x) := \sum_{j=1}^{n} \frac{d_{\infty}(x, X \setminus U_j)}{\nu(x)} v_j.
\]

We would like to set \( \psi := r \circ \psi' : Y \to X \) where \( \psi' : Y \to I^\infty \) is the function

\[
\psi' \left( \sum_{j=1}^{n} \alpha_j v_j \right) := \sum_{j=1}^{n} \alpha_j x_j.
\]

In addition, we would like to define \( h : X \times [0, 1] \to X \) by setting

\[
h(x, t) := r \left( (1 - t)\psi'(\varphi(x)) + tx \right).
\]

In order to complete the proof it now suffices to show that:

(a) \( \psi'(Y) \subset U \);

(b) for all \( x \in X \), the line segment \( \text{conv}(\{x, \psi'(\varphi(x))\}) \) is contained in \( U \);

(c) for all \( x \in X \), \( r \left( \text{conv}(\{x, \psi'(\varphi(x))\}) \right) \subset B_\varepsilon(x) \).

Picking an arbitrary \( y = \sum_{j=1}^{n} \alpha_j v_j \in |K| \), let \( i_0, \ldots, i_p \) be the indices \( j \) such that \( \alpha_j > 0 \). The definition of \( K \) implies the existence of a point \( z \in \bigcap_{k=0}^{p} U_{i_k} \). Without loss of generality assume that \( \eta(x_{i_0}) \geq \eta(x_{i_1}), \ldots, \eta(x_{i_p}) \), and let \( B := B_{2\eta(x_{i_0})}(x_{i_0}) \). Then

\[
d_{\infty}(z, x_{i_k}) < \eta(x_{i_k}) \leq \eta(x_{i_0})
\]

for all \( k = 0, \ldots, p \), so \( x_{i_k} \in B \) for all \( k \). The definition of \( \eta(x_{i_0}) \) implies that

\[
B \subset r^{-1}(B_{\varepsilon/2}(x_{i_0}) \cap X) \subset U.
\]

In addition, \( B \) is convex, so \( U \) contains \( \psi'(y) = \sum_{k=0}^{p} \alpha_{i_k} x_{i_k} \). This establishes (a).

Now suppose that \( y = \varphi(x) \) for some \( x \in X \). The definition of \( \varphi \) implies that \( i_0, \ldots, i_p \) are precisely the indices \( j \) such that \( x \in U_j \). Note that \( B := B_{2\eta(x_{i_0})}(x_{i_0}) \supset B_{\eta(x_{i_0})}(x_{i_0}) = U_{i_0} \), so \( B \) contains \( x \), and since it is convex, it must contain the line segment between \( x \) and \( \psi'(\varphi(x)) \). Thus (b) holds. Moreover, \( B \subset U \), so

\[
h(x, t) \in r(B) \subset B_{\varepsilon/2}(x_{i_0}) \subset B_\varepsilon(x)
\]

for all \( 0 \leq t \leq 1 \). This establishes (c) and completes the proof. \( \blacksquare \)
10.2 Approximation by Functions

The following result is a central element of our “homology-free” development of advanced fixed point theory.

**Theorem 10.2.1** Let $X$ and $Y$ be ANRs with $X$ compact. Suppose that $D \subset U \subset X$ with $D$ compact and $U$ open. If $F : U \to Y$ is an upper semicontinuous contractible (and compact) valued correspondence, and $W \subset U \times Y$ is a neighborhood of $\text{Gr}(F)$, then there is a continuous function $f : D \to Y$ with $\text{Gr}(f) \subset W$.

The remainder of the section is devoted to the proof, beginning with the following special case, which is taken, with some modifications, from [MC74]. The reader should be warned that, at least relative to most of the rest of the book, this proof is hard. Perhaps it would be an insult to suggest that you might be tempted to skip it, but at the same time I would recommend that you tackle it when you are well rested, fortified with your favorite caffeinated beverage, and have ample time, not only to puzzle out the difficult parts, but also to savor the satisfaction of having learned something complex and challenging.

**Proposition 10.2.2** Let $K = (V,C)$ be a simplicial complex, let $Z$ be an open subset of a convex subset $S$ of a normed linear space $T$, and let $F : |K| \to Z$ be an upper semicontinuous contractible valued correspondence. Then any neighborhood of $\text{Gr}(F)$ contains the graph of a continuous function $f : |K| \to Z$.

**Proof:** Fix a metric $d$ on $|K|$, and let the metric on $|K| \times Z$ be given by letting the distance between $(x,z)$ and $(x',z')$ be $d(x,x') + \|z - z'\|$. Since $|K|$ is compact, the graph of $F$ is also compact, so it suffices to show that for any $\varepsilon > 0$ there is $f : |K| \to Z$ with $\text{Gr}(f) \subset B_{\varepsilon}(\text{Gr}(F))$. Fix $\varepsilon > 0$ small enough that, for all $(x,z) \in \text{Gr}(F)$, the ball in $S$ of radius $\varepsilon$ around $z$ is contained in $Z$.

We proceed by induction on the dimension $m$ of $K$. The claim is clearly correct when $K$ is 0-dimensional. Suppose that $K$ is $m$-dimensional, and that the claim is known to be true for $(m-1)$-dimensional complexes.

For each $x \in |K|$ choose:

(a) a contraction $\varphi_x : F(x) \times [0,1] \to F(x)$;

(b) a number $\delta_x$ small enough that

$$\|\varphi_x(z,t) - \varphi_x(z',t')\| < \varepsilon/2$$
whenever \((z, t), (z', t') \in F(x) \times [0, 1]\) with \(|z - z'| + |t - t'| < \delta_x; \)

(c) a number \(\gamma_x \leq \varepsilon / 2\) small enough that \(B_{\gamma_x}(F(x')) \subset B_{\delta_x / 3}(F(x))\) for all \(x' \in B_{\gamma_x}(x).\)

(The existence of such a \(\gamma_x\) follows from the upper semicontinuity of \(F\).)

Let \(\varphi\) be the constant value of \(\varphi_x(\cdot, 1).\)

Choose \(x_1, \ldots, x_N\) such that

\[
B := \{B_{\gamma_{x_1}/2}(x_1), \ldots, B_{\gamma_{x_N}/2}(x_N)\}
\]

is a covering of \(|K|\). If \(\alpha > 0\) is sufficiently small, for all \(x \in |K|\) there is some \(i = 1, \ldots, N\) such that \(B_{\alpha}(x) \subset B_{\gamma_{x_i}/2}(x_i)^1\). It suffices to prove the claim with \(K\) replaced by the simplicial complex obtained by applying barycentric subdivision any finite number of times. Therefore Proposition ??? implies that we may assume that each simplex of \(K\) is contained in some \(B_{\gamma_{x_i}/2}(x_i).\) Let \(K'\) be the \((m - 1)\)-skeleton of \(K\). That is, \(K'\) is obtained from \(K\) by removing all \(m\)-simplices. The induction hypothesis implies that there is a continuous function \(f' : |K'| \to Z\) with

\[
\text{Gr}(f') \subset B_{\frac{\alpha}{\min\{\gamma_{x_1}, \ldots, \gamma_{x_N}\}}}(\text{Gr}(F\big|_{|K'|})).
\]

Take an arbitrary \(m\)-simplex \(L \in K\). To conclude the proof it suffices to show that \(f'|_{\partial L}\) can be extended to a continuous function \(f : L \to Z\) in such a manner that \(\text{Gr}(f) \subset B_{\frac{\alpha}{\gamma_{x_i}}}(\text{Gr}(F)).\) Fix an \(i\) such that \(L \subset B_{\gamma_{x_i}}(x_i)\).

For any \(y \in \partial L\) there is a point

\[
(y', z') \in B_{\gamma_{x_i}/2}(y, f'(y)) \cap \text{Gr}(F\big|_{K'}).
\]

In particular, \(y' \in B_{\gamma_{x_i}/2}(y) \subset B_{\gamma_{x_i}}(x_i)\) and \(f'(y) \in B_{\gamma_{x_i}}(F(y'))\), so \(f'(y) \in B_{\delta_{x_i}/3}(F(x_i)).\) That is, for each \(y \in \partial L\) there exists

\[
w \in B_{\delta_{x_i}/3}(f'(y)) \cap F(x_i).
\]

Let \(\overline{x}\) denote the barycenter of \(L\). For every \(x \in L \setminus \{\overline{x}\},\)

\[
(y(x), t(x)) \in \partial L \times [0, 1]
\]

\(^1\text{This is easily proved by contradiction: otherwise there is a sequence } \{x_j\} \text{ such that for each } j \text{ there is no } i \text{ with } B_{\gamma_{x_j}}(x_j) \subset B_{\gamma_{x_i}/2}(x_i), \text{ but any limit point of such a sequence cannot be in any } B_{\gamma_{x_i}/2}(x_i).\)
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will denote the “polar” coordinates of \( x \), i.e., the unique \( (y, t) \in \partial L \times [0, 1] \) such that \( x = t\overline{x} + (1 - t)y \). These coordinate functions are continuous on \( L \setminus \{\overline{x}\} \). Define

\[
A_1 := \{ x \in L : t(x) \leq 1/4 \}, \quad A_2 := \{ x \in L : 1/4 \leq t(x) \leq 1/2 \}, \\
A_3 := \{ x \in L : 1/2 \leq t(x) \}.
\]

By continuity there is \( 0 < \lambda \) such that

\[
\| f'(y(x)) - f'(y(x')) \| < \frac{\delta_{x_i}}{3}
\]

for all \( x, x' \in L \) such that \( \|x - x'\| < \lambda \) and \( t(x), t(x') \leq 1/2 \). It is intuitively obvious that there is a simplicial subdivision of \( L \) for which \( A_2 \) is a subcomplex, but an explicit construction or argument to this effect would require a major digression, so we will take it as given, leaving a demonstration as an exercise for the reader. Possibly after repeated barycentric subdivision, we may assume that we have a simplicial subdivision of \( A_2 \) whose mesh is less than \( \lambda \). Let \( \hat{V} \) be the vertices of this subdivision.

By construction, for every \( x \in A_1 \),

\[
B_{\delta_{x_i}/3}(f'(y(x))) \cap F(x_i) \neq \emptyset.
\]

Hence for every \( v \in \hat{V} \) we can pick

\[
w(v) \in B_{\delta_{x_i}/3}(f'(y(x))) \cap F(x_i)
\]

and set

\[
f''(v) = \varphi_{x_i}(w(v), 4t(v) - 1).
\]

Define \( f'' : A_2 \to Z \) by linear extension on each of the simplices of the subdivision of \( A_2 \). Finally, let \( f : L \to Z \) be given by

\[
f(x) := \begin{cases} (1 - 4t(x))f'(y(x)) + 4t(x)f''(\frac{1}{4}\overline{x} + \frac{3}{4}y(x)), & x \in A_1 \\ f''(x), & x \in A_2, \\ \overline{x}, & x \in A_3. \end{cases}
\]

The first and second cases agree on \( A_1 \cap A_2 \), and the second and third cases agree on \( A_2 \cap A_3 \), so \( f \) is well defined and continuous.

It remains to show that \( \text{Gr}(f) \subset B_{\gamma_i}(\text{Gr}(F)) \). We have \( L \subset B_{\gamma_{x_i}}(x_i) \), and \( \gamma_i \leq \varepsilon/2 \). Therefore it suffices to show that, for all \( x \in L \), \( f(x) \in B_{\varepsilon/2}(F(x_i)) \), since then it follows that

\[
(x, f(x)) \in B_{\varepsilon}(\{x_i\} \times F(x_i)).
\]
Consider $x \in A_1$. Fix a particular vertex $\overline{v} \in \hat{V}$ of the simplex containing $\frac{1}{4}+\frac{3}{4}y(x)$. For each other vertex $v$ of this simplex we have $d(v, \overline{v}) < \lambda$, so that $\|f'(y(v)) - f'(y(\overline{v}))\| < \delta_x/3$ and thus $w(v) \in B_{\delta_x/2}(w(\overline{v}))$ and $\|f''(v) - f''(\overline{v})\| < \varepsilon/2$. Since $f''(\frac{1}{4}+\frac{3}{4}y(x))$ is a convex combination of the points $w(v)$ for the various vertices of this simplex, we have

$$f''(\frac{1}{4}+\frac{3}{4}y(x)) \in B_{\varepsilon/2}(w(\overline{v})).$$

We also have $d(y(x), y(\overline{v})) < \delta_x/3$, so that

$$\|f'(y(x)) - w(\overline{v})\| \leq \|f'(y(x)) - f'(y(\overline{v}))\| + \|f'(y(\overline{v})) - w(\overline{v})\|
< \delta_x/3 + \delta_x/3 = 2\delta_x/3 < \varepsilon/2.$$ 
Since $f(x)$ is a convex combination of $f'(y(x))$ and $f''(\frac{1}{4}+\frac{3}{4}y(x))$, it is in $B_{\varepsilon/2}(w(\overline{v})) \subset B_{\varepsilon/2}(F(x_i))$.

Now suppose that $x \in A_2$. Fixing a vertex $\overline{v}$ of the simplex of the subdivision of $A_2$ that contains $x$, for each other vertex $v$ we have $d(v, \overline{v}) < \lambda$, so that $\|f'(y(v)) - f'(y(\overline{v}))\| < \delta_x/3$ and thus $w(v) \in B_{\delta_x/2}(w(\overline{v}))$ and $\|f''(v) - f''(\overline{v})\| < \varepsilon/2$. Since $w(\overline{v}) \in F(x_i)$ and $f''(x)$ is a convex combination of the points $f''(v)$ for the vertices $v$ of the simplex of the subdivision of $A_2$ that contains $x$, it follows that

$$f(x) = f''(x) \in B_{\varepsilon/2}(w(\overline{v})) \subset B_{\varepsilon/2}(F(x_i)).$$

If $x \in A_3$, then $f(x) = \overline{x} \in F(x_i)$. The proof is complete.

At this point we pause for a technical result.

**Lemma 10.2.3** Suppose that $(S, d)$ is a metric space, $T$ is a topological space, $F : S \to T$ is an upper semicontinuous compact valued correspondence, and $W \subset S \times T$ be a neighborhood of $\text{Gr}(F)$. Then there is $\varepsilon > 0$ and a neighborhood $W'$ of $\text{Gr}(F)$ such that $(s, t) \in W$ whenever $(s', t) \in W'$ and $d(s, s') < \varepsilon$.

**Proof:** In view of the definition of the product topology, for every $(s, t) \in \text{Gr}(F)$ there exist $\delta_{(s, t)} > 0$ and an open neighborhood $V_{(s, t)} \subset T$ of $t$ such that

$$B_{\delta_{(s, t)}}(s) \times V_{(s, t)} \subset W.$$
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Consider a particular \( s \in S \). Since \( F(s) \) is compact there exist \( t_1, \ldots, t_K \) such that \( V_{(s,t_1)}, \ldots, V_{(s,t_K)} \) is a cover of \( F(s) \). Letting

\[
\delta_s := \min\{\delta_{(s,t_1)}, \ldots, \delta_{(s,t_K)}\} \quad \text{and} \quad V_s := V_{(s,t_1)} \cup \cdots \cup V_{(s,t_K)},
\]

we have

\[
\{s\} \times F(s) \subset B_{\delta_s}(s) \times V_s \subset W.
\]

Replacing \( \delta_s \) with a smaller number if need be, we may assume without loss of generality that \( F(s') \subset V_s \) for all \( s' \in B_{\delta_s}(s) \).

Choosing \( s_1, \ldots, s_M \) such that \( \bigcup_m B_{\delta_{s_m}/2}(s_m) = S \), let \( \varepsilon := \min\{\delta_{s_m}/2\} \), and set

\[
W' := \bigcup_m B_{\delta_{s_m}/2}(s_m) \times V_{s_m}.
\]

The remainder of the proof of Theorem 10.2.1 generalizes Proposition 10.2.2 in two ways, first replacing the simplicial complex of its domain space with an ANR, then generalizing by allowing an ANR (rather than an open subset of a convex subset of a normed linear space) as the range space. The proof of the first generalization uses the Domination Theorem.

**Proposition 10.2.4** Let \( X \) be a compact ANR, and let \( Z \) be an open subset of a convex subset of a locally convex space. Suppose \( D \subset U \subset X \) where \( D \) is compact and \( U \) is open. If \( F : U \to Z \) is an upper semicontinuous contractible (and compact) valued correspondence and \( W \subset U \times Z \) is a neighborhood of \( \operatorname{Gr}(F) \), then there is a continuous \( f : D \to Z \) with \( \operatorname{Gr}(f) \subset W \).

**Proof:** Since \( U \) may be replaced with a smaller neighborhood of \( D \) whose closure is contained in \( U \), no generality is lost in assuming that \( W \subset U \times Z \) is a neighborhood of \( \operatorname{Gr}(F) \) in \( U \times Z \) where \( F : U \to Z \) is an upper semicontinuous contractible valued correspondence. We may now choose \( \varepsilon_1 > 0 \) and \( W' \subset W \) as per Lemma 1. Let \( \varepsilon_2 > 0 \) be small enough that the open \( \varepsilon_2 \) ball around \( D \) is contained in \( U \), and let \( \varepsilon := \min\{\varepsilon_1, \varepsilon_2\} \). Let \( K \) be a polyhedron that \( \varepsilon \)-dominates \( X \) by virtue of the maps \( \varphi : X \to |K| \) and \( \psi : |K| \to X \). Since \( d_X(x, \psi(\varphi(x))) < \varepsilon \) for all \( x \), we have \( \varphi(D) \subset \psi^{-1}(U) \).

Since \( \varphi(D) \) is compact and \( \psi^{-1}(U) \) is open, after suitably fine subdivision we can find a subcomplex \( J \subset K \) with \( \varphi(D) \subset J \subset \psi^{-1}(U) \). Proposition 1 implies the existence of a continuous \( g : J \to Z \) with \( \operatorname{Gr}(g) \subset (\psi \times \text{Id}_Z)^{-1}(W') \).

Let \( f = g \circ \varphi|_J \). For \( (x, f(x)) \in \operatorname{Gr}(f) \) we have \( (\psi(\varphi(x)), f(x)) \in W' \) and \( d_X(x, \psi(\varphi(x))) < \varepsilon \), so \( (x, f(x)) \in W \).
Proof of Theorem 10.2.1: As per Proposition 9.4.2, fix a retraction \( r: Z \to Y \) with inclusion \( i: Y \to Z \) where \( Z \) is a relatively open subset of a convex subset of a locally convex space. Then \( (\text{Id}_U \times r)^{-1}(W) \) is a neighborhood of \( \text{Gr}(i \circ F) \) in \( U \times Z \), so Lemma 10.2.4 implies the existence of a continuous \( g: U \to Z \) with \( \text{Gr}(f) \subset (\text{Id}_U \times r)^{-1}(W) \). Letting \( f = r \circ g \), we clearly have \( \text{Gr}(f) \subset W \).

Many of the important consequences of this result will be developed later, in Chapter ??, but we can already derive a version of the Eilenberg-Montgomery theorem:

**Theorem 10.2.5** If \( X \) is a compact ANR with the fixed point property, then any upper semicontinuous contractible valued correspondence \( F: X \to X \) has a fixed point.

**Proof:** In the last result let \( Y = X \) and \( D = U = X \). Endow \( X \) with a metric \( d_X \). For each \( j = 1, 2, \ldots \) let

\[
W_j := \{ (x', y') \in X \times X : d_X(x, x') + d_X(y, y') < 1/j \}
\]

for some \( (x, y) \in \text{Gr}(F) \), let \( f_j: X \to X \) be a continuous function with \( \text{Gr}(f_j) \subset W_j \), let \( z_j \) be a fixed point of \( f_j \), and let \( (x'_j, y'_j) \) be a point in \( \text{Gr}(F) \) with \( d_X(x'_j, z_j) + d_X(y'_j, z_j) < 1/j \). Passing to convergent subsequences, we find that the common limit of the sequences \( \{x'_j\}, \{y'_j\}, \) and \( \{z_j\} \) is a fixed point of \( F \).

### 10.3 CONTINUATION

A point to bear in mind is the conjecture at the end of the JME paper, which constitutes a barrier to completely implementing the program. Probably we should work on the index theory chapter after this, as a way of getting the story fully straightened out before going on to the game theory.

**Problems**

**Problem 10.1:** Give a direct proof that Kinoshita's example (Section 9.1) is not an ENR. (That is, do not simply point out that if it was an ENR, Theorem ?? would imply that it had the fixed point property.)