Lecture 8
Social Welfare Functions

I. Framework
A. We are considering a function $W : \mathbb{R}^I \to \mathbb{R}$ that is viewed as encoding some system of preferences over allocations of utilities to the individuals $1, \ldots, I$.

II. Basic Properties of Social Welfare Functions
A. Nonpaternalism
B. Paretian
C. Symmetry
D. Concavity

III. Examples of Social Welfare Functions
A. Utilitarian: $W(u) = \sum_{i=1}^{I} \beta_i u_i$.
B. Rawlsian: $W(u) = \min\{u_1, \ldots, u_I\}$.
C. Generalized utilitarian: $W(u) = \sum_{i=1}^{I} g_i(u_i)$.
   1. The constant elasticity case is $g_1 = \ldots = g_I = g_{\rho}$ where $g_{\rho}(u) = (1 - \rho)u^{1-\rho}$.

IV. Compensation Principle
A. The issue here is whether to undertake a project that would change the set of feasible utility vectors $U$.
B. The compensation principle is the idea that the project should be undertaken if the new feasible set contains points that Pareto dominate the old outcome, even if the new outcome does not Pareto dominate the old outcome.
V. Pairwise Independence Condition

A. Framework: $X$ is a set of alternatives, $\mathcal{U} = \mathbb{R}^X$ is the set of possible utility functions, $\mathcal{R}$ is the set of complete transitive relations on $X$, and $F: \mathcal{U}^I \rightarrow \mathcal{R}$ is a social welfare functional.

B. Paretian property.

C. We ask when $F$ can be induced by some $W: \mathbb{R}^I \rightarrow \mathbb{R}$. We say that $F$ satisfies the pairwise independence condition if, whenever $\bar{u}$ and $\bar{u}'$ are two vectors of utility functions such that $(\bar{u}_1(x), \ldots, \bar{u}_I(x)) = (\bar{u}'_1(x), \ldots, \bar{u}'_I(x))$ and $(\bar{u}_1(y), \ldots, \bar{u}_I(y)) = (\bar{u}'_1(y), \ldots, \bar{u}'_I(y))$, then $xF(\bar{u})y$ if and only if $xF(\bar{u}')y$.

**Proposition:** Suppose that

(a) $|X| \geq 3$;

(b) $F$ satisfies the pairwise independence condition;

(c) $F$ is Paretian,

Then there is a complete transitive ordering $\succeq$ of $\mathbb{R}^I$ such that, for all $\bar{u}$ and $x, y \in X$, $xF(\bar{u})y$ if and only if $(\bar{u}_1(x), \ldots, \bar{u}_I(x)) \succeq (\bar{u}'_1(y), \ldots, \bar{u}'_I(y))$.

VI. Invariance to Common Changes of Origin or Units

A. We say that $F$ is invariant with respect to common changes of origins if $F(\bar{u} - \alpha \mathbf{e}) = F(\bar{u})$ for all $\bar{u}$ and all $\alpha \in \mathbb{R}$. We say that $F$ is invariant with respect to common changes of units if $F(\beta \bar{u}) = F(\bar{u})$ for all $\bar{u}$ and all $\beta > 0$.

We say that $F$ is invariant with respect to common cardinal transformations if both conditions hold.

**Proposition:** Suppose that $F$ is generated by a continuous and increasing welfare function $W$, and $F$ is invariant with respect to common changes of origins. Then there is a real valued function $g$ whose domain is $\{ u \in \mathbb{R}^I : \bar{u} = 0 \}$ (where $\bar{u} := (1/I) \sum_i u_i$) such that $F$ is generated by $W'(u) := \bar{u} + g(u - \bar{u} \mathbf{e})$. If $F$ is also invariant with respect to common changes of units, then $g$ is homogeneous of degree one.

VII. Invariance to Individual Changes of Origin or Units

A. We say that $F$ is invariant with respect to individual changes of origins if
\[ F(\tilde{u}-(\alpha_1, \ldots, \alpha_I)) = F(\tilde{u}) \] for all \( \tilde{u} \) and all \( (\alpha_1, \ldots, \alpha_I) \in \mathbb{R}^I \). We say that \( F \) is \textit{invariant with respect to individual changes of units} if \( F(\beta_1 \tilde{u}_1, \ldots, \beta_I \tilde{u}_I) = F(\tilde{u}) \) for all \( \tilde{u} \) and all \( \beta_1, \ldots, \beta_I > 0 \). We say that \( F \) \textit{does not allow interpersonal comparisons of utility} if both conditions hold.

\textbf{Proposition:} Suppose that \( F \) is generated by a continuous and increasing welfare function, and \( F \) is invariant with respect to individual changes of origins. Then \( F \) can be generated by a purely utilitarian social welfare function \( W(u) = \sum_i \beta_i u_i \). If \( F \) is also invariant with respect to individual changes of units, then \( F \) is dictatorial.