Economics 8103  
Microeconomic Theory  
Spring-1 2004

Lecture 5  
Bayesian Games and Auction Theory

I. Bayesian Nash Equilibria of Games of Incomplete Information

A. Very often it is natural to assume that the “physically” possible strategies are commonly known, but that the agents have private information concerning the payoffs.

1. In principle it is not only each agent’s information about payoffs that is relevant, but also the agents’ information about each others’ information about payoffs, the agents’ information about each others’ information about each others’ information about payoffs, and so on. The space of such infinite hierarchies of beliefs can be described and analyzed, but it is a highly infinite dimensional thing employing advanced analytical tools.

2. One of the important contributions of the analysis just mentioned is to justify the simpler “types” model of Harsanyi (which we study here) by showing that “types” models are (in the relevant topology) dense in the set of all possible infinite hierarchy models.

3. A Bayesian game (in strategic form) is a tuple
\[ G = (I, (S_1, \ldots, S_n), (\Theta_1, \ldots, \Theta_n), (u_1, \ldots, u_n), F) \]

where \( I = \{1, \ldots, n\} \) is the set of agents, for each \( i \in I \), \( S_i \) and \( \Theta_i \) are
nonempty finite sets of pure strategies and types respectively, and

\[ u_i : S_1 \times \ldots \times S_n \times \Theta_1 \times \ldots \times \Theta_n \to \mathbb{R} \]

is a payoff function, and \( F \in \Delta(\Theta_1 \times \ldots \times \Theta_n) \) is a Bayesian prior distribution.

a. To simplify notation we let

\[ S = S_1 \times \ldots \times S_n \quad \text{and} \quad \Theta = \Theta_1 \times \ldots \times \Theta_n. \]

B. A (mixed) decision rule for player \( i \) is a function \( \sigma_i : \Theta_i \to \Delta(S_i) \).

1. The expected utility resulting from a vector \( \sigma = (\sigma_1, \ldots, \sigma_n) \) of decision rules is

\[ \sum_{\theta \in \Theta} \left( \sum_{s \in S} \left( \prod_{i \in I} \sigma_i(s_i | \theta_i) \right) u_i(s, \theta) \right) F(\theta). \]

2. A Bayesian Nash equilibrium is a vector of decision rules with the property that each agent is maximizing expected utility.

3. Since we may write the expected utility of agent \( i \) as

\[ \sum_{\theta_i \in \Theta_i} \left( \sum_{\theta_{-i} \in \Theta_{-i}} \left( \sum_{s \in S} \left( \prod_{j \in I} \sigma_j(s_j | \theta_j) \right) u_i(s, \theta) \right) F(\theta_{-i} | \theta_i) \right) F_i(\theta_i) \]

(here \( F_i \in \Delta(\Theta_i) \) is the marginal distribution on \( \Theta_i \) and \( F(\theta_{-i} | \theta_i) \) is the conditional probability of \( \theta_{-i} \) given \( \theta_i \)) we see that maximization for agent \( i \) is a matter of maximizing the expected utility of each of her types separately.

4. In this sense, a Bayesian Nash equilibrium is merely a Nash equilibrium of the strategic form game in which each type is regarded as a different agent.

C. The purification theorem of Harsanyi explains how mixed strategy equilibria (as perceived by “the social scientist”) might occur without any agent consciously randomizing. If actual payoffs differ from the payoffs as described by
the model according to small and independently distributed perturbations, with each agent’s perturbation known by that agent, but not by others, then a pure equilibrium of the underlying Bayesian game will resemble a mixed equilibrium of the model.

D. An Example

1. Let $\Theta_1 = \{H, D\}$, let $\Theta_2$ be a singleton, and let $F(H) = .7$ and $F(D) = .3$. Let $S_1 = \{G, B\}$, and $S_2 = \{T, I\}$, and let the payoffs be

$$
\begin{pmatrix}
T \\
G \\
B
\end{pmatrix} = \begin{pmatrix}
((8, 8), 5) & ((5, 5), 4) \\
((7, 9), 0) & ((0, 2), 3)
\end{pmatrix}
$$

where the first pair in each entry gives the payoffs of the types $H$ and $D$ respectively.

2. The interpretation is that agent 1 is a supplier who can choose to deliver either good or bad products to agent 2, the customer. Agent 1 is either honest (type $H$) in which case it is a dominant strategy to supply good products, or dishonest (type $D$) in which case the gain resulting from getting away with supplying bad goods outweighs whatever feelings of guilt this entails. Agent 2 can either trust agent 1 or inspect the goods before accepting them, which is costly for all concerned.

3. In any Bayesian Nash equilibrium the honest type will supply good products. If the dishonest type also supplies good products then the best response for agent 2 is to never inspect, in which case the dishonest type does better to supply bad products. On the other hand if the dishonest type always supplies bad products, then the best response of agent 2 is to always inspect, since the gain of 3 with probability .3 outweighs the loss of 1 with probability .7, and if inspection is ex-
pected the dishonest type prefers to deliver good products. The only Bayesian Nash equilibrium has agent 2 mixing between trust and inspection in a way that makes the dishonest type indifferent, which means that inspection will occur with probability .25. The dishonest type must mix in a way that leaves agent 2 indifferent, which means that the posterior probability of good products must be 3/4.

II. Introduction to Auctions

A. Auctions are an ancient institution.

1. One would like to explain this stability as a consequence of optimality.

B. Many markets utilize auctions.

1. Mineral rights.
2. Slaves.
3. Art.
4. Flowers.
5. Wine.
7. State owned businesses.

C. Questions.

1. Equilibrium strategies and outcomes.
2. Optimality.
   a. Pareto.
   b. For the seller.
3. Do auction outcomes resemble market outcomes?
4. Aggregation of information.

III. Types of Auction.

A. Dutch – the price descends continuously until a bidder yells “stop”.
B. First price sealed bid – the highest bidder pays his or her bid.
C. Second price sealed bid – the highest bidder pays the second highest bid.
D. English – bids are announced until there is a bid that no one wants to top.

IV. Models of Valuation

A. Each agent observes a signal $\theta_i$, which we now assume to be a real number.
B. Each player $i = 1, \ldots, I$ has a value $u_i(\theta_1, \ldots, \theta_I)$ for the object and is risk neutral with respect to money, so that the payoff is either $u_i(\theta_1, \ldots, \theta_I) - p_i$ if agent $i$ wins or $-p_i$ if the agent does not, where $p_i$ is the payment from the agent to the auctioneer.

1. In most auctions $p_i = 0$ when agent $i$ does not win, but some attention has been given to “all pay auctions” in which each agent pays her bid regardless of whether she wins.

C. The distribution of signal vectors $(\theta_1, \ldots, \theta_I)$ is assumed to be symmetric.
D. There is $u : \mathbb{R}^I \to \mathbb{R}$ such that for all $i$ and $\theta$,

\[ u_i(\theta) = u(\theta_i, \{\theta_j\}_{j \neq i}) \]

1. **Assumption:** $u$ is nonnegative, continuous, and nondecreasing.

2. **Assumption:** $\mathbb{E}(u_i) < \infty$.

E. **Example:** The Independent Private Values Model.

1. $u_i(\theta_1, \ldots, \theta_I) = \theta_i$.

2. $\theta_1, \ldots, \theta_I$ are independent and identically distributed.

F. **Example:** The Mineral Rights Model.

1. The unknown “true value” $v$ is the same for everyone.

2. Conditional on $v$, $\theta_i$ and $\theta_j$ are statistically independent. (For example it might be the case that $\theta_i = v + \epsilon_i$ where $\epsilon_1, \ldots, \epsilon_I$ are i.i.d. normal random variables.)
3. In this setting one has the so-called “winner’s curse” – the expected value of
the object conditional on $\theta_i$ may be much higher than the expected value
conditional on $\theta_i$ and $\theta_i = \max\{\theta_j\}$.

V. Symmetric Equilibrium in Second Price Auctions.

A. We assume that all valuations are monetary and that all agents are risk neutral.

B. We assume that agents $j \neq i$ are following the same bidding strategy that is a
monotonic increasing function $\mathbf{b}: \theta_j \mapsto \mathbf{b}(\theta_j)$, and we analyze the behavior of
agent $i$.

1. Let $Y_1 = \max_{j \neq i} \theta_j$ be the top order statistic of the other agents’ signals.

2. Agent $i$’s problem is $\max \mathbf{E}[(u_i - \mathbf{b}(Y_1)) \cdot 1_{\{\mathbf{b}(Y_1) < b\}} | \theta_i]$. 

C. Set $v(\theta, y) = \mathbf{E}[u_i | \theta_i = \theta, Y_1 = y]$. Let $\mathbf{b}^*(\theta) = v(\theta, \theta)$.

**Theorem:** If the model of valuation is such that $v$ is nondecreasing in both variables,
then $(\mathbf{b}^*, \ldots, \mathbf{b}^*)$ is an equilibrium.

**Proof:** The consequences of, say, increasing the bid over $\mathbf{b}^*(\theta_i)$ to $b$ is that one wins in
situations where one would have lost before, namely when $Y_1$ is between $\theta_i$ and $\mathbf{b}^{*-1}(b)$.
Since, for such $Y_1$, $v(\theta_i, Y_i) \leq v(\mathbf{b}^{*-1}(b), \mathbf{b}^{*-1}(b)) = b$, this is not good. A similar argument
shows that one does not do better by bidding less that $\mathbf{b}^*(\theta_i)$.

VI. Symmetric Equilibrium in First Price Auctions.

A. Again we wish to determine when $(\mathbf{b}^*, \ldots, \mathbf{b}^*)$ is an equilibrium. Assume $\mathbf{b}^*$ is
increasing and differentiable.

B. Assume all $j \neq i$ play $\mathbf{b}^*$ while bidder $i$ observes $\theta_i$ and bids $b$. Her expected
payoff is

$$\Pi_i(b; \theta_i) = E[(u_i - b) \cdot 1_{b^*(Y_i) < b}] | \theta_i]$$

$$= E[E[(u_i - b) \cdot 1_{b^*(Y_i) < b}] | \theta_i, Y_i] | \theta_i]$$

$$= E[(v(\theta_i, Y_i) - b) \cdot 1_{b^*(Y_i) < b}] | \theta_i]$$

$$= \int_{b^*(\theta_i)}^{b^{*\prime}(\theta_i)} (v(\theta_i, \alpha) - b) f_{Y_i}(\alpha | \theta_i) d\alpha$$

Here $f_{Y_i}(\cdot | \theta_i)$ is the density of $Y_i$ given $\theta_i$. Let $F_{Y_i}(\cdot | \theta_i)$ be the cumulative distribution function.

C. If $b = b^*(\theta)$ is optimal then (after some work) the first order condition yields the differential equation

$$b'^{\prime}(\theta) = (v(\theta, \theta) - b^*(\theta)) \left[ \frac{f_{Y_i}(\theta | \theta)}{F_{Y_i}(\theta | \theta)} \right]$$

1. Necessarily $v(\theta, \theta) - b^*(\theta)$ is nonnegative for all $\theta$, since otherwise one is bidding when one prefers to not win.

2. Let $\theta$ be the lowest possible signal. If $v(\theta, \theta) - b^*(\theta) > 0$ then one does better by replacing $b^*(\theta)$ with $b^*(\theta) + \varepsilon$, so

$$b^*(\theta) = v(\theta, \theta).$$

VII. The Revenue Equivalence Theorem.

A. In the independent private values model the expected revenue in the second price auction is the expectation of the second order statistic of $\theta_1, \ldots, \theta_I$.

1. If $F(\cdot)$ is the c.d.f. of each $\theta_i$, the c.d.f. of $\max\{\theta_1, \ldots, \theta_I\}$ is $F(\cdot)^I$.

2. If one knows $\theta_i$ and that $\theta_j \leq \theta_i$, the c.d.f. of $\theta_j$ is $F(\cdot) / F(\theta_i)$.

B. In the independent private values model with $\theta$ distributed uniformly on $[0, 1]$, $b^*(\theta) = \frac{I-1}{I} \theta$ is the symmetric equilibrium strategy. Therefore the expected revenue is $\frac{I-1}{I}$ times the expectation of the first order statistic of $\theta_1, \ldots, \theta_I$. 
C. The revenue equivalence theorem asserts that, in the independent private values model, all auction forms have the same expected revenue for the seller.

1. Let $e(p)$ be the expected payment of the least cost strategy that wins the object with probability $p$.

2. Let $p^*(\theta)$ be the equilibrium probability of winning the object as a function of $\theta$.

3. The marginal condition for optimization is

$$\frac{d[p\theta - e(p)]}{dp} \bigg|_{p=p^*(\theta)} = 0 \quad \text{or} \quad \theta = e'(p^*(\theta)), \quad \text{so} \quad \frac{d[e(p^*(\theta))]}{d\theta} = \theta \cdot \frac{dp^*(\theta)}{d\theta}.$$ 

4. We now have the following computation

$$e(p^*(\theta)) = e(p^*(0)) + \int_0^\theta \left( \frac{d[e(p^*(\beta))]}{d\beta} \right) \, d\beta \quad \text{or} \quad e(p^*(\theta)) = e(p^*(0)) + \int_0^\theta w \, dp^*(w).$$ 

5. All the auction forms discussed above give the object to the agent with the highest valuation, so they all have the same equilibrium $p^*$ function and the analysis above (which is basically a matter of applying the incentive compatibility constraints from the revelation principle) shows that consequently they have the same $e$ function.