Economics 8104  
Microeconomic Theory  
Spring 2004

Lecture 4  
Kakutani’s Fixed Point Theorem: A New Proof

I. Introduction

A. Kakutani’s fixed point theorem is the most important mathematical result underlying economic theory.

1. It is the “engine” of the existence proofs for Nash equilibrium and the existence of competitive equilibrium in general equilibrium theory.
2. An understanding of fixed point theory contributes to the analysis of at least two issues that are important in both of these branches of mathematical economics.
   a. “Learning” or otherwise converging to an equilibrium.
   b. Stability of equilibria.

B. Proofs of Kakutani’s theorem are typically not studied except in advanced courses in mathematical economics.

1. Speaking personally, I find this morally offensive.
   a. Any mathematician would think it ridiculous to “train” new researchers without teaching the proofs of the key results in their fields.
2. Of course the reason for this neglect is that (until now!) all proofs of Kakutani’s fixed point theorem suffered from two defects.
   a. They are long and technical.
b. They require the development of mathematical concepts of limited interest in economics.

C. We will study a new proof recent discovered by Rabee Tourky (of the University of Melbourne) and myself.

1. In comparison with other proofs, it is brief and simple.

2. Turning the tables, game theory is applied to topology: the basis of the argument is a special case of the Nash equilibrium existence theorem that is easy to prove, but which implies the general Kakutani theorem by a simple argument.

II. Imitation Games

A. A finite two person normal form game is specified by nonempty finite sets $S$ and $T$ of pure strategies and payoff functions $u, v : S \times T \to \mathbb{R}$.

1. A point $(\sigma, \tau) \in \Delta(S) \times \Delta(T)$ is a Nash equilibrium if $u(\sigma, \tau) \geq u(\sigma', \tau)$ for all $\sigma' \in \Delta(S)$ and $v(\sigma, \tau) \geq v(\sigma, \tau')$ for all $\tau' \in \Delta(T)$.

2. We say that the game is an imitation game if $S = T$ and

$$v(s, t) := \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

B. There are two main steps:

1. Prove that every imitation game has a Nash equilibrium.

2. Use this to prove Kakutani’s theorem.

III. The Proof of Kakutani’s Theorem

A. We reverse the logical order of the two steps. Assume that it is already known that every imitation game has a Nash equilibrium.

B. Our goal is:
Kakutani’s Fixed Point Theorem: If $C \subset \mathbb{R}^n$ is nonempty, compact, and convex, and $F: C \to C$ is a correspondence whose values $F(x)$ are convex and whose graph

$$\{ (x, y) : x \in C, y \in F(x) \}$$

is closed, then $F$ has a fixed point: that is, $x^* \in F(x^*)$ for some $x^* \in C$.

C. Let $\{\varepsilon^r\}$ be a sequence of positive numbers converging to zero. For each $r$:

1. let $D^r$ be a finite subset of $C$ that is $\varepsilon^r$-dense: $C \subset \bigcup_{x \in D^r} B_{\varepsilon^r}(x)$;
2. let $f^r : D^r \to C$ be a function with $f^r(x) \in F(x)$ for each $x \in D^r$;
3. define an imitation game $(S^r, T^r; u^r, v^r)$ by setting $S^r = T^r = D^r$ and letting $u^r, v^r : S^r \times T^r \to \mathbb{R}$ be the functions

$$u^r(s, t) := -\|s - f^r(t)\|^2 \quad \text{and} \quad v^r(s, t) := \begin{cases} 1, & t = s, \\ 0, & t \neq s; \end{cases}$$

4. let $(\sigma^r, \tau^r)$ be a Nash equilibrium of $(S^r, T^r; u^r, v^r)$;
5. let $\omega^r := \sum_{t \in D^r} \tau^r(t) f^r(t)$.

Claim 1: The support of $\tau^r$ is a subset of the support of $\sigma^r$. (This is obviously true for any Nash equilibrium of any imitation game.)

Claim 2: The support of $\sigma^r$ is a subset of the set of points in $S^r = D^r$ closest to $\omega^r$.

To see this observe that $\sum_{t \in D^r} \tau^r(t) (\omega^r - f^r(t)) = 0$, so for any $x \in D^r$ we have

$$u^r(x, \tau^r) = \sum_{t \in D^r} \tau^r(t) u^r(x, t) = -\sum_{t \in D^r} \tau^r(t) \| (x - \omega^r) + (\omega^r - f^r(t)) \|^2$$

$$= -\sum_{t \in D^r} \tau^r(t) \langle x - \omega^r, x - \omega^r \rangle - 2 \sum_{t \in D^r} \langle x - \omega^r, \tau^r(t) (\omega^r - f^r(t)) \rangle$$

$$- \sum_{t \in D^r} \tau^r(t) \langle \omega^r - f^r(t), \omega^r - f^r(t) \rangle$$

$$= -\|x - \omega^r\|^2 - \sum_{t \in D^r} \tau^r(t) \|\omega^r - f^r(t)\|^2.$$ 

D. Finishing the proof.

1. Passing to a subsequence if necessary, we may assume that $\omega^r \to x^*$. 

3
2. Fixing $\delta > 0$, choose $\gamma > 0$ small enough that $F(x) \subset B_\delta(F(x^*))$
whenever $\|x - x^*\| < \gamma$.
3. For sufficiently large $r$ we have $\|\omega^r - x^*\| < \gamma/2$ and $\varepsilon^r < \gamma/2$.
4. This implies that $t \in B_{\gamma/2}(\omega^r) \subset B_{\gamma}(x^*)$, so that $f^r(t) \in B_\delta(F(x^*))$,
for all $t \in \text{supp}(\tau^r)$.
   a. Since $F(x^*)$ is convex, so is $B_\delta(F(x^*))$, so it contains $\omega^r$.
5. We now have
\[
x^* = \lim_{r \to \infty} \omega^r \in \bigcap_{\delta > 0} B_\delta(F(x^*)) = F(x^*).
\]

IV. Genericity for Two Person Games

A. Let nonempty finite sets of pure strategies $S$ and $T$ be given.
   1. When $A \subset S$ and $B \subset T$ are nonempty, $s^* \in A$, and $u \in \mathbb{R}^{S \times T}$, define
      \[M_{A,B}^s(u)\]
to be the $(\#A - 1) \times \#B$ matrix with rows corresponding to elements of $A \setminus \{s^*\}$, columns corresponding to elements of $B$, and
      $(s, t)$-entry $u(s, t) - u(s^*, t)$.
   2. Let $M_{A,B}^{s^*}(u)$ be the $\#A \times \#B$ matrix obtained from $M_{A,B}^s(u)$ be
      appending a row of 1's at the bottom.
      a. We will think of $M_{A,B}^{s^*}(u)$ as a linear transformation with domain
         $\mathbb{R}^B$.
      b. If $(\sigma, \tau)$ is a Nash equilibrium for $(S, T; u, v)$ with $\text{supp}(\sigma) = A$
         and $\text{supp}(\tau) = B$, then $M_{A,B}^{s^*}(u)\tau = (0, \ldots, 0, 1)$.
      c. Define
         \[K_{A,B}^{s^*}(u) := M_{A,B}^{s^*}(u)^{-1}(0, \ldots, 0, 1)\]
   B. An affine subspace of $\mathbb{R}^m$ is a set of the form
      \[K = x + V = \{ x + v : v \in V \}\]
where $V$ is a linear subspace of $\mathbb{R}^m$.
1. The dimension of $K$ is the dimension of $V$.

2. Let $G$ be the set of $u \in \mathbb{R}^{S \times T}$ such that for all $A$, $s^*$, and $B$ as above,
$$K_{A,B}^{s^*}(u) = (\#B - \#A) \text{-dimensional.}$$

Lemma 1: The interior of $G$ is dense in $\mathbb{R}^{S \times T}$.

Proof: Let $A$, $s^*$, and $B$ be as above. When $\#A \leq \#B$, elementary linear algebra shows that $K_{A,B}^{s^*}(u)$ is $(\#B - \#A)$-dimensional if and only if the $M_{A,B}^{s^*}(u)$ has full rank. Let $\hat{M}_{A,B}^{s^*}(u)$ be the matrix obtained from $M_{A,B}^{s^*}(u)$ by appending $(0, \ldots, 0, 1)$ as an additional column. If $\#A > \#B$, $K_{A,B}^{s^*}(u)$ is $(\#B - \#A)$-dimensional (i.e., empty) if $\hat{M}_{A,B}^{s^*}(u)$ has full rank. A matrix has full rank if and only if the determinant of at least one of its maximal minors is nonzero, so the set of $u$ for which $M_{A,B}^{s^*}(u)$ or $\hat{M}_{A,B}^{s^*}(u)$ has full rank is a finite union of sets defined by the nonvanishing of some nontrivial polynomial in the variables $u(s,t)$, hence open and dense. (There are numerous methods of showing that if a polynomial function vanishes on an open set, then the polynomial’s coefficients are all zero, so that it vanishes everywhere.) Thus the interior of $G$ is open and dense because it is the intersection of a finite collection of open dense sets. \(\blacksquare\)

C. We now define and analyze certain important sets.

1. For $u : S \times T \to \mathbb{R}$ and $\tau \in \Delta(T)$ let $PBR_u(\tau) := \arg\max_s u(s, \tau)$ be the set of pure best responses to $\tau$.

2. Let $A \subset S$ and $B \subset T$ be nonempty.

3. When $\#A = \#B$ we define
$$n_{A,B}(u) := \{ \tau \in \Delta(B) : A = PBR_u(\tau) \}.$$

4. When $\#A = \#B - 1$ we define
$$e_{A,B}(u) := \{ \tau \in \Delta(B) : A \subset PBR_u(\tau) \}.$$

   a. Clearly these sets are convex, and $e_{A,B}(u)$ is closed.
Lemma 2: Let $A \subset S$ and $B \subset T$ be nonempty. If $u \in G$, then:

(a) If $\#A > \#B$, then \{ $\tau \in \Delta(B) : A \subset PBR_u(\tau)$ \} is the null set;

(b) If $\#A = \#B$, then $n_{A,B}(u)$ is either empty or zero dimensional;

(c) If $\#A = \#B - 1$, then $e_{A,B}(u)$ is either empty or one dimensional.

Proof: Fix $u \in G$, and choose some $s^* \in A$. Then

$$\{ \tau \in \Delta(B) : A \subset PBR_u(\tau) \} \subset K_{A,B}^{s^*}(u),$$

so (a) and (b) follow directly from the definition of $G$, and (c) can fail to hold only if $e_{A,B}(u)$ is zero dimensional. To produce a contradiction suppose that $\#A = \#B - 1$ and $e_{A,B}(u)$ is zero dimensional, i.e., a singleton, say $e_{A,B}(u) = \{ \tau_0 \}$. Let $\overline{A} = PBR_u(\tau_0)$ and $\overline{B} := \text{supp}(\tau_0)$. Then $A \subset \overline{A}$ and $\overline{B} \subset B$. Since $\tau_0 \in K_{A,B}^{s^*}(u)$, the latter set is nonempty, so (a) implies that $\#A \leq \#B$, whence

$$\#(\overline{A} \setminus A) + \#(B \setminus \overline{B}) \leq 1.$$

Since $u \in G$, $K_{A,B}^{s^*}(u)$ is a one dimensional affine subspace of $R^B$. We cannot have $\overline{A} = A$ and $\overline{B} = B$, since then the definitions of $e_{A,B}(u)$, $\overline{A}$, and $\overline{B}$ imply that $e_{A,B}(u)$ contains a neighborhood of $\tau_0$ in $K_{A,B}^{s^*}(u)$, contrary to our assumption that it is a singleton. Suppose $s \in \overline{A} \setminus A$. Then the linear function $u(s,\tau) - u(s^*,\tau)$ must be positive for $\tau$ in one of the open half lines in $K_{A,B}^{s^*}(u)$ determined by $\tau_0$ and negative for $\tau$ in the other, so

$$K_{A,B}^{s^*}(u) \cap \{ \tau \in R^B : u(s,\tau) \leq u(s^*,\tau) \}$$

is a ray emanating from $\tau_0$. But in this case $e_{A,B}(u)$ must include a neighborhood of $\tau_0$ in this ray. A similar contradiction is attained when $t \in B \setminus \overline{B}$:

$$K_{A,B}^{s^*}(u) \cap \{ \tau \in R^B : \tau(t) \geq 0 \}$$

is a ray emanating from $\tau_0$, and $e_{A,B}(u)$ includes some neighborhood of $\tau_0$ in this ray.  \[\blacksquare\]
V. The Lemke-Howson Algorithm

A. We now prove that every imitation game has a Nash equilibrium.

1. The argument uses a simplified version of the Lemke-Howson algorithm.

   a. The Lemke-Howson is a method for computing a single equilibrium of a two player game.

2. Assume that \((S, T; u, v)\) is an imitation game.

   a. The set \(S = T\) can be denoted in two ways, and in each instance below we choose the notation that corresponds most closely to the role the set is playing in the argument.

B. We will assume that \(u \in G\).

1. To see that it suffices to establish the claim in this case, observe that for any \(u\), Lemma 1 implies the existence of a sequence \(\{u^j\}\) in \(G\) that converges to \(u\). For each \(j\) let \((\sigma^j, \tau^j)\) be a Nash equilibrium for \((u^j, v)\).

2. Passing to a subsequence if necessary, we may assume that \((\sigma^j, \tau^j) \rightarrow (\sigma, \tau)\).

3. Since the graph of the Nash equilibrium correspondence is closed, \((\sigma, \tau)\) is an equilibrium for \((u, v)\).

4. Henceforth we will assume that \(u \in G\).

C. Observe that if a set \(e_{A,B}(u)\) is nonempty, it is (by (c) of Lemma 2) a line segment, and its endpoints are the unique elements of sets of the forms \(n_{A \cup \{s\}, B}(u)\) and \(n_{A, B \setminus \{t\}}(u)\).

1. For a nonempty \(A \subset S\) let \(m_A\) be the unique element of \(n_{A,A}(u)\) when the latter set is nonempty.

2. We wish to show that there is such an \(A\), since then there is a Nash equilibrium \((\ell_A, m_A)\) where \(\ell_A \in \Delta(S)\) is the mixed strategy that
assigns probability $1/\#A$ to each element of $A$ and 0 to each element of $S \setminus A$.

D. Fix a particular pure strategy $t^* \in T$.

1. For a nonempty $A \subset S$ let $e_A := e_{A, A \cup \{t^*\}}(u)$ if this set is nonempty, and for $s \in A$ let $n_{A,s}$ be the unique element of $n_{A, (A \setminus \{s\}) \cup \{t^*\}}(u)$ when this set is nonempty.

2. Usually $n_{A,s}$ is an endpoint of precisely two sets $e_{A'}$, namely $e_A$ and $e_{A \setminus \{s\}}$.
   a. The one exception occurs when $A = \{s\}$ is a singleton, in which case $e_{A \setminus \{s\}}$ is undefined, so that $n_{\{s\}, s}$ is an endpoint of $e_{\{s\}}$ and no other $e_{A'}$.

3. Observe that when it is defined, $n_{\{s\}, s}$ is in $\Delta(\{\{s\}\} \cup \{t^*\}) = \{t^*\}$, so that $s$ must be a best response to $t^*$.
   a. From (a) of Lemma 2 we know that there is only one pure best response to $t^*$, which we denote by $s_0$.

E. Completing the Argument

1. One possibility is that $s_0 = t^*$, in which case we are done because $m_{\{t^*\}}$ is defined.

2. Otherwise (the degenerate probability measure assigning all mass to) $t^*$ is $n_{\{s_0\}, s_0}$, and $n_{\{s_0\}, s_0}$ is an endpoint of $e_{\{s_0\}}$ and no other $e_{A'}$.
   a. If the other endpoint of $e_{\{s_0\}}$ is of the form $m_A$ we are done.

2. Otherwise it is $n_{A_1, s_1}$ for some $s_1 \in A_1 \subset S$.

3. Proceeding inductively, if $n_{A_j, s_j}$ is an endpoint of the $e_A$ and $e_{A'}$, where $n_{A_{j-1}, s_{j-1}}$ is the other endpoint of $e_A$, then either the other endpoint of $e_{A'}$ is of the form $m_{A''}$ or it is $n_{A_{j+1}, s_{j+1}}$ for some $s_{j+1} \in A_{j+1} \subset S$.

4. Since this process cannot continue forever, and it cannot return to any
$n_{A_j, s_j}$ it has previously visited, it must eventually arrive at a point of the form $m_A$.

5. The proof that every imitation game has a Nash equilibrium is now complete.

\[ \begin{array}{c}
\text{b} \\
\{c\} \cup \{a, c\} \\
\{a, c\} \cup \{c\} \\
\{a, c\} \cup \{a\} \\
\{a, b\} \cup \{c\} \\
\{a, b\} \cup \{b\} \\
\{a\} \cup \{a, b\}
\end{array} \]

F. In discussions of algorithms of this sort, the “bookkeeping” involved in keeping track of which pure strategies are optimal and/or unused is conventionally described as a matter of “attaching labels.”

1. For $\tau \in \Delta(T)$ the relevant sets of labels are

$$\lambda(\tau) := PBR_\mu(\tau) \quad \text{and} \quad \mu(\tau) = \{ t \in T : \tau(t) = 0 \}.$$ 

2. We say that $\tau$ is completely labelled if

$$\lambda(\tau) \cup \mu(\tau) = S = T.$$ 

In this event $\tau = m_{\lambda(\tau)}$. 

9
3. Relative to the chosen $t^* \in T$, we say that $\tau$ is $t^*$-almost completely labelled if

$$T \setminus \{t^*\} \subseteq \lambda(\tau) \cup \mu(\tau).$$

4. Figure 1 shows the three paths emanating from the three possible starting points of the algorithm.

a. Here $S = T = \{a, b, c\}$, and for each pure strategy $z$ the underlined strategy $\underline{z}$ indicates the region in which $z$ is the unique pure best response.

b. Each point $p$ of the forms $m_A$ and $n_{A,s}$ (relative to some choice of $t^*$) is labelled with $\lambda(p) \cup \mu(p)$. 