Economics 8103  
Microeconomic Theory  
Spring 2004-1  

Lecture 3  
Nash Equilibrium  

I. Introduction  

A. A Nash equilibrium of a normal form game is a vector of mixed strategies with the property that each agent’s strategy is optimal if his or her expectations about the behavior of others are given by the strategy vector.  

B. Our discussion of this concept has the following parts.  

1. We give a precise definition and state Nash’s existence theorem.  

2. The concept has numerous interesting philosophical interpretations.  

   We will discuss the main ones.  

3. We will begin to develop the skill of solving a game for its set of Nash equilibria.  

4. The extension of the model to games of incomplete information will be described.  

II. Definition and Existence  

A. Let a normal form game $N = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ be given.  

1. Recall the formula for agent $i$’s expected payoff:  

   $$u_i(\sigma) = \sum_{s \in S} \left( \prod_{j=1, \ldots, n} \sigma_j(s_j) \right) \cdot u_i(s) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}).$$  

2. Note that $u_i$ is a continuous function on $\Sigma = \Delta(S_1) \times \cdots \times \Delta(S_n)$.  

3. We say that $\tau_i \in \Delta(S_i)$ is a best response for agent $i$ to the strategy vector $\sigma$ (actually $\sigma_{-i}$) if $u_i(\tau_i, \sigma_{-i}) \geq u_i(\rho_i, \sigma_{-i})$ for all $\rho_i \in \Delta(S_i)$.  

a. Let $BR_i(\sigma)$ be the set of best responses for $i$ to $\sigma$.

b. Our formula above shows that $\tau_i \in BR_i(\sigma)$ if and only if $\tau_i$ assigns all probability to pure strategies that are best responses.

c. Geometrically this means that $BR_i(\sigma)$ is the face of $\Delta(S_i)$ spanned by the pure best responses for $i$ to $\sigma$.

d. In particular $BR_i(\sigma)$ is convex.

4. For $\sigma \in \Sigma$ let $BR(\sigma) = BR_1(\sigma) \times \cdots \times BR_n(\sigma)$.

**Definition:** A Nash equilibrium of the game $N$ is a mixed strategy vector $\sigma^* \in \Sigma$ such that $\sigma^* \in BR(\sigma^*)$.

B. Existence.

1. The most important theorem in game theory is:

**Theorem:** (Nash) Every normal form game has a Nash equilibrium.

2. For arbitrary sets $X$ and $Y$ a correspondence $F : X \rightarrow Y$ is a "rule" that associates a nonempty set $F(x) \subset Y$ with each point $x \in X$.

3. If $X$ and $Y$ are metric spaces, and $F(x)$ is compact for all $x \in X$, $F$ is said to be upper semicontinuous if, for each $x_0 \in X$ and each open set $V$ containing $F(x_0)$, there is an open set $U$ containing $x_0$ such that $F(x) \subset V$ for all $x \in U$.

   a. If $X$ and $Y$ are compact, then $F$ is upper semicontinuous if and only if the graph

   $$\{(x, y) : x \in X, y \in F(x)\}$$

   is closed.

3. We note the following properties of the best response correspondence $BR : \Sigma \rightarrow \Sigma$.

   a. $\Sigma$ is nonempty, compact, and convex.
b. For each $\sigma \in \Sigma$, $BR(\sigma)$ is convex since it is the cartesian product of convex sets.

c. The graph of $BR$ is closed.

i. Specifically, suppose $\{\sigma^m\}$ and $\{\tau^m\}$ are sequences in $\Sigma$ with $\tau^m \in BR(\sigma^m)$ for all $m$, $\sigma^m \to \sigma$, and $\tau^m \to \tau$. If $\tau \notin BR(\sigma)$ then there would be some $i$ and some $\rho \in \Delta(S_i)$ with $u_i(\rho, \sigma_{-i}) > u_i(\tau, \sigma_{-i})$. Since $u_i$ is continuous this would imply that $u_i(\rho, \sigma^m_{-i}) > u_i(\tau^m, \sigma^m_{-i})$ and $\tau^m \notin BR(\sigma^m)$ for large $m$, contrary to our assumption.

4. Nash’s theorem now follows from the following mathematical result which is very important in economics.

**Kakutani’s Fixed Point Theorem:** If $C \subset \mathbb{R}^n$ is nonempty, compact (i.e. closed and bounded), and convex, and $F: C \to C$ is a correspondence whose values $F(x)$ are convex and whose graph is closed, then $F$ has a fixed point: that is, $x^* \in F(x^*)$ for some $x^* \in C$.

5. Kakutani proved this theorem by showing that every neighborhood of the graph of $F$ in $C \times C$ contains the graph of a continuous function $F: C \to C$, applying Brouwer’s fixed point theorem, and using a limiting argument.

**Brouwer’s Fixed Point Theorem:** If $C \subset \mathbb{R}^n$ is nonempty, compact, and convex and $f: C \to C$ is a continuous function, then $f$ has a fixed point: $x^* = f(x^*)$ for some $x^* \in C$.

6. This is a deep result of twentieth century mathematics.

a. It will not be proved here.

b. One way to think about this is as follows. If you have two pieces of paper one on top of the other (with edges lined up), and you take the top one, crumple it up, and place it on the bottom one, then there will be some point in the top piece that
is over the same point it was over originally.

III. Interpretations of Nash Equilibrium

A. The one shot interpretation of a game provides no foundation for the Nash equilibrium concept, since it provides no explanation of the equality of an agent’s strategy and the expectations of others concerning his behavior.

1. Although Nash’s theorem was recognized as an important result right away, there was also considerable uneasiness about the idea.

2. Some people tried to argue that Nash equilibrium might be a reasonable notion for a game with a unique equilibrium.
   a. This idea seems flawed since there are games with unique equilibria whose sets of rationalizable strategies are much larger.
   b. It has led people to search for single valued solution concepts that assign a particular Nash equilibrium to each game.
      i. This has not been particularly successful.
      ii. Such a procedure should assign a symmetric equilibrium to a symmetric game, but in symmetric games the asymmetric equilibria are often more plausible.

B. A cautious attitude would be to say that if there is an “obvious” way to play a game then it must be a Nash equilibrium, since otherwise it would be irrational, but that in many games there will be no obvious way to play.

C. A more modern interpretation is to view a game as an interaction that occurs repeatedly in society, but always between agents who do not know each other’s past behavior (so that “reputation building” is impossible) and who will not encounter each other again (so that the current play is not influenced by expectations of rewards or punishments later on).

1. Each time the game occurs the agents form their expectations about the behavior of others by looking at how the game has been played in
the past.

2. A Nash equilibrium can then be described as a “potentially self-reproducing pattern of strategic expectations.”

3. When we think about this idea a little more carefully we see that it is not really “the same” game that is repeated, but “similar” games.

   a. Thus this interpretation depends on an understanding within society of which games fall into which “similarity classes.”

D. Focal points.

1. When there are several equilibria, particularly in games that are predominantly matters of cooperation rather than conflict, people seem to try to find some strategy that is “focal” in a way that is best illustrated by example.

   2. Suppose you and a friend wanted to meet in New York City at a particular time on a particular day, but you had not agreed on a location and could not communicate. Where would you go? When asked many people say they would go to the clock in Grand Central Station.

3. Focal points seem to be an example of the interpretation offered above. In games of “coordination” (a “similarity class”) the expectation in our society is that one will look for “prominent” strategies.

E. Nash equilibria as self enforcing agreements.

1. A quite different use of the Nash equilibrium concept arises in situations in which agents can talk to each other, and form agreements as to how to play the game, prior to the beginning of the game, but there is no enforcement mechanism providing independent incentives for compliance with agreements.

   a. In order to be a credible agreement in such an environment a
strategy vector must be a Nash equilibrium.

b. As we will see, some Nash equilibria are not properly regarded as equilibria, so some Nash equilibria may not be credible agreements.

2. This idea plays an important role in international affairs. It is generally agreed that any acceptable arms control agreement must be “verifiable,” the point being that the expectation that violations will be detected and countered deters violations.

3. The concept of “honor” can be viewed as a device for escaping from the limitations of self enforcing agreements. If one is deterred from violating agreements by the fear of losing one’s reputation or honor, then a larger set of agreements will be credible.

IV. Solving for the Set of Nash Equilibria

A. Solving a game for its set of Nash equilibria is a matter of systematically working through a process of elimination.

1. One can think of an equilibrium as having two parts.

   a. Each agent’s strategy has a support, the set of pure strategies played with positive probability.

   b. The actual mixed strategies assigned to the supports.

2. One has the following crude procedure for solving any game.

   a. Enumerate all possible vectors of supports for the agents.

   b. For each vector of supports find all vectors of mixed strategies with the given supports such that each agent is indifferent between all elements of its support. This is a matter of solving a system of polynomial equations and inequalities.

   c. Eliminate from this set all mixed strategy vectors in which some agent would prefer to play a pure strategy not in his
3. In practice this procedure is speeded up by grouping closely related possibilities. One has chains of reasoning like:

We begin by considering the possibility that there are equilibria in which agent 1 plays A. In this case agent 2 will not assign any probability to strategies other than b and c, so there are three possibilities: (i) agent 2 assigns all probability to b; (ii) agent 2 mixes between b and c; (iii) agent 2 assigns all probability to c. We consider these cases in turn .

(a) In this type of analysis it is important to be patient and well organized.

(b) Skill and strategic insight often allow one to find a way to structure the cases so that many possibilities are eliminated at once and the analysis is relatively short.

B. Consider the general $2 \times 2$ game:

\[
\begin{array}{ccc}
2 & s_2 & t_2 \\
1 \\
\hline
s_1 & (a_1, a_2) & (b_1, c_2) \\
t_1 & (c_1, b_2) & (d_1, d_2)
\end{array}
\]

1. A totally mixed Nash equilibrium must solve the system:

\[
\sigma_1(s_1)a_2 + \sigma_1(t_1)b_2 = \sigma_1(s_1)c_2 + \sigma_1(t_1)d_2, \\
\sigma_1(s_1) + \sigma_1(t_1) = 1, \\
\sigma_2(s_2)a_1 + \sigma_2(t_2)b_1 = \sigma_2(s_2)c_1 + \sigma_2(t_2)d_1, \\
\sigma_2(s_2) + \sigma_2(t_2) = 1.
\]

(b) Solving this pair of systems of two equations in two unknowns yields

\[
(\sigma_1(s_1), \sigma_1(t_1)) = \left( \frac{d_2 - b_2}{a_2 - b_2 - c_2 + d_2}, \frac{a_2 - c_2}{a_2 - b_2 - c_2 + d_2} \right) \quad \text{and} \\
(\sigma_2(s_2), \sigma_2(t_2)) = \left( \frac{d_1 - b_1}{a_1 - b_1 - c_1 + d_1}, \frac{a_1 - c_1}{a_1 - b_1 - c_1 + d_1} \right).
\]
c. Observe that, rather paradoxically, agent 1’s strategy is determined by agent 2’s payoffs and vice versa.