Economics 7250
Advanced Mathematical Techniques for Economics
Second Semester 2014

Lecture 1
Course Overview

I. Why Mathematics for Economics?

A. Mathematics allows economics models to be expressed precisely and rigorously.
   1. Issues related to the properties of a model can be settled unambiguously. (Disputes are now between models.)
   2. Economics can take advantage of advances in mathematics, and explore issues suggested by mathematical considerations.

B. What are our objectives?
   1. Primarily, a well founded understanding of the central mathematical tools applied in economics.
   2. In comparison with earlier courses, the emphasis will be on understanding the logical framework. Knowing algorithms and being able to compute will continue to be important, but ideally these skills should flow naturally from a deeper understanding.

C. Why the theme of optimization?
   1. Optimization is the mathematics related to rational choice, which is the fundamental hypothesis of theoretical social science.
   2. Optimization forms a coherent mathematical theory.
II. The General Form of Optimization Problems

A. Our goal in the rest of this lecture is to establish some terminology, and to get a picture, in the form of a simple particular example, of how the main ideas in the course work.

B. **Example 1:** The notation

\[
\max_{x,y \geq 0, \ x + y \leq 1} x \cdot y
\]

specifies a problem in which the objects of choice are pairs of numbers \((x, y)\) satisfying the constraints \(x \geq 0, \ y \geq 0, \) and \(x + y \leq 1,\) and the goal is to make the quantity \(x \cdot y\) as large as possible.

C. The general form of an *optimization problem* is

\[
\max_{x \in D} f(x).
\]

1. \(D\) is called the *domain*, the *choice set*, the *feasible set*, or the *constraint set*.
2. \(f\) is called the *objective function*, or the *utility function*.
3. \(f(D) = \{f(x) : x \in D\}\) is called the *image of \(f\)*, or the *set of attainable values*.
4. A point \(x^* \in D\) is a *solution*, or an *optimum*, or a *maximizer*, if \(f(x^*) \geq f(x)\) for all \(x \in D\).
   a. The notation for the set of solutions is

\[
\text{argmax}_{x \in D} f(x),
\]

which stands for the set of *arguments that maximize \(f\).*
5. As a matter of convention, in economics it is less common to consider minimization problems which have the general form

\[
\min_{x \in D} f(x).
\]

For such problems \( f \) is usually called the loss function, and “minimizer” is a synonym for “solution.”

III. Central Issues

A. Existence.

1. For a variety of reasons it can be important to know whether a solution exists. In particular, propositions of the form “all solutions have property \( P \)” can be true, and quite silly, if the set of solutions is empty.

2. We will develop a way of showing that Example 1 has a solution using only the facts that:
   a. \( f(x, y) = x \cdot y \) is a continuous function of \( x \) and \( y \).
   b. \( D = \{ (x, y) : x, y \geq 0, x + y \leq 1 \} \) is compact, meaning, practically, that it is both closed and bounded.

3. Of course another way to demonstrate the existence of an optimizer is to find one.

B. Necessary Conditions for an Optimum

1. From calculus we know that if \( t^* \) is a solution of

\[
\max_{a < t < b} g(t),
\]

where \( g : (a, b) \to \mathbb{R} \) is differentiable, then \( g'(t^*) = 0 \) and \( g''(t^*) \leq 0 \).
2. We will pay particular attention to the extension of these results to higher dimensional domains.

3. If Example 1, the solution \((\frac{1}{2}, \frac{1}{2})\) lies on the boundary of the feasible set. The Lagrangean theorem asserts, in this instance, that there is some \(\lambda \geq 0\) such that the \textit{first order conditions}

\[
\frac{\partial L}{\partial x}(\frac{1}{2}, \frac{1}{2}; \lambda) = 0 \quad \text{and} \quad \frac{\partial L}{\partial y}(\frac{1}{2}, \frac{1}{2}; \lambda) = 0
\]

(which generalize \(g'(t^*) = 0\) above) are satisfied by the Lagrangean function

\[
L(x, y; \lambda) = x \cdot y + \lambda(1 - x - y).
\]

Intuitively, the term \(\lambda(1 - x - y)\) acts as a penalty for being on the wrong side of the constraint that is just sufficient to make you not “want” to violate the constraint.

C. Sufficient Conditions for Optimality

1. In Example 1, if we make the reasonable assumption that \(x+y = 1\), we obtain the problem

\[
\max_{0 \leq x \leq 1} x(1-x).
\]

2. A set \(S \subset \mathbb{R}^n\) is \textit{convex} if, for every \(x, x' \in S\), every point on the line segment between \(x\) and \(x'\) is contained in \(S\).

3. A function \(f : S \rightarrow \mathbb{R}\) is \textit{concave} if the the region

\[
\{ (x, v) : v \leq f(x) \} \subset S \times \mathbb{R}
\]

lying below the graph of the function is convex.
a. **Exercise:** Graph the function \( f(x) = x(1-x) \) and convince yourself that it is convex.

4. If \( g : (a, b) \to \mathbb{R} \) is concave and \( g'(t^*) = 0 \), then \( t^* \) is a solution of

\[
\max_{a < t < b} g(t).
\]

This principle generalizes to higher dimensions.

D. Dependence on Parameters.

1. Example 1 could arise as a particular instance of the problem:

\[
\max_{x, y \geq 0, p_x x + p_y y \leq I} xy.
\]

2. In economic applications, in particular, it is often important to know how the solution \((x^*(p_x, p_y, I), y^*(p_x, p_y, I))\) changes as \(p_x, p_y, \) and \(I\) vary.

3. The equations that are usually used to characterize the solutions of optimization problems, which assert that certain derivatives are zero, do not give the solution directly as a function of the parameters, but instead give equations, for instance of the form

\[
F(p_x, p_y, I, x^*(p_x, p_y, I), y^*(p_x, p_y, I)) = (0, 0) \in \mathbb{R}^2,
\]

that characterize the solutions *implicitly.*

4. The *implicit function theorem* gives conditions under which the solution functions \(x^*\) and \(y^*\) are “nicely behaved,” and shows how to compute their partial derivatives in terms of the partial derivatives of \(F\).
E. Computation.

1. There is a practical aspect, namely learning to compute the solution in particular instances. This is basically a matter of generating the appropriate set of equations and solving it.

2. For many optimization problems computation is important from a more theoretical standpoint.
   a. Algorithms that automate computations of optima are very important practically.
   b. Many of the central problems in the theory of computational complexity are optimization problems.