I. Introduction

A. The seller of an auction usually has the power to choose the auction mechanism, and of course would like to maximize expected revenue.

1. In the symmetric case the form of the optimal auction will turn out to be quite simple: a second price sealed bid auction with a reserve price.

2. The analysis is quite sophisticated, with some important concepts.

B. The Environment.

1. There are \( I \) risk neutral bidders whose valuations for the object are independent (but not identically distributed) random variables \( v_1, \ldots, v_I \in [0, 1] \).

   a. For each \( i \) let \( F_i : [0, 1] \rightarrow [0, 1] \) be the cumulative distribution function of \( v_i \). We assume that \( F_i \) is \( C^1 \), and we let \( f_i \) be the density.

2. A direct selling mechanism can be characterized by functions 

   \[ p_1, \ldots, p_I : [0, 1]^I \rightarrow [0, 1] \]

   giving the probabilities of winning the object at each vector of valuations and functions

   \[ c_1, \ldots, c_I : [0, 1]^I \rightarrow \mathbb{R} \]

   giving the costs charged at each vector of valuations.

   a. Let \( v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I) \) denote a vector of valuations for the agents other than \( i \).
b. Let
\[ p_i(r_i) = \int_0^1 \cdots \int_0^1 p_i(r_i, v_{-i}) \left( \prod_{j \neq i} f_j(v_j) \right) dv_j \]
and
\[ c_i(r_i) = \int_0^1 \cdots \int_0^1 c_i(r_i, v_{-i}) \left( \prod_{j \neq i} f_j(v_j) \right) dv_j \]
be the probability of winning and the expected cost of reporting \( r_i \).

c. Let
\[ u_i(r_i, v_i) = p_i(r_i) v_i - c_i(r_i) \]
be agent \( i \)'s expected utility if her type is \( v_i \) and she reports \( r_i \).

3. Incentive compatibility requires that \( u_i(v_i, v_i) \geq u_i(r_i, v_i) \) for all \( r_i \) and \( v_i \).
4. Individual rationality requires that \( u_i(v_i, v_i) \geq 0 \) for all \( v_i \).
5. Feasibility requires that \( \sum_i p_i(v) \leq 1 \) for all \( v \in [0,1]^I \).

II. A Characterization of Incentive Compatibility

**Theorem:** The direct selling mechanism \((p_i(\cdot), c_i(\cdot))_{i=1}^I\) is incentive compatible if and only if, for every bidder \( i \):

(a) \( \overline{p}_i \) is nondecreasing.

(b) \( \overline{c}_i(v_i) = \overline{c}_i(0) + \overline{p}_i(v_i) v_i - \int_0^{v_i} \overline{p}_i(t) \, dt \) for all \( v_i \in [0,1] \).

**Proof:** First suppose that the mechanism is incentive compatible. For any \( i, r_i, \) and \( v_i \) incentive compatibility gives \( u_i(r_i, v_i) \leq u_i(v_i, v_i) \) and \( u_i(v_i, r_i) \leq u_i(r_i, r_i) \), or

\[ p_i(r_i) v_i - c_i(r_i) \leq p_i(v_i) v_i - c_i(v_i) \quad \text{and} \quad p_i(v_i) r_i - c_i(v_i) \leq p_i(r_i) r_i - c_i(r_i) \]

Adding these and cancelling costs gives

\[ p_i(r_i) v_i + p_i(v_i) r_i \leq p_i(v_i) v_i + p_i(r_i) r_i \]

Rearranging gives

\[ 0 \leq (p_i(v_i) - p_i(r_i))(v_i - r_i) \]

which shows that \( p_i \) is nondecreasing.
Since \( u_i(r_i, v_i) \) is maximized, for a given \( v_i \), by setting \( r_i = v_i \), we must have

\[
0 = \frac{\partial u_i}{\partial r_i}(v_i, v_i) = \overline{p}_i'(v_i) v_i - \overline{c}_i'(v_i).
\]

Therefore the fundamental theorem of calculus and integration by parts give

\[
\overline{c}_i(v_i) - \overline{c}_i(0) = \int_0^{v_i} \overline{c}_i'(t) \, dt = \int_0^{v_i} \overline{p}_i'(t) t \, dt = \overline{p}_i(v_i) v_i - \int_0^{v_i} \overline{p}_i(t) \, dt.
\]

This establishes (b).

Now suppose that (a) and (b) hold. For any \( i, v_i, \) and \( r_i \) we apply (b), obtaining

\[
u_i(r_i, v_i) = \overline{p}_i(r_i) v_i - \overline{c}_i(r_i) = \overline{p}_i(v_i) v_i - \left[ \overline{c}_i(0) + \overline{p}_i(r_i) r_i - \int_{0}^{r_i} \overline{p}_i(t) \, dt \right]
\]

\[
= -\overline{c}_i(0) + \int_{r_i}^{v_i} \overline{p}_i(t) \, dt + \int_{0}^{v_i} \overline{p}_i(t) \, dt - \int_{r_i}^{v_i} \overline{p}_i(t) \, dt
\]

\[
= -\overline{c}_i(0) + \int_{0}^{v_i} \overline{p}_i(t) \, dt - \int_{r_i}^{v_i} (\overline{p}_i(t) - \overline{p}_i(r_i)) \, dt.
\]

From (b) we have \(-\overline{c}_i(0) + \int_{0}^{v_i} \overline{p}_i(t) \, dt = \overline{p}_i(v_i) v_i - \overline{c}_i(v_i) = u_i(v_i, v_i)\), and the integral on the right hand side is nonnegative because \( \overline{p}_i \) is nondecreasing. Therefore \( u_i(r_i, v_i) \leq u_i(v_i, v_i) \), which is to say that the mechanism is incentive compatible. ■

III. Revenue Equivalence

A. The following result can be seen as a consequence of the general result from the last lecture, but we reprove it anyway. The intuition comes out of the characterization of incentive compatibility above, because \( \overline{c}_i(v_i) \) can be computed from \( \overline{c}_i(0) \) and \( \overline{p}_i(\cdot) \).

**Theorem:** If two incentive compatible mechanisms have the same probability assignment functions, and for each \( i \) the two mechanisms give agent \( i \) with valuation 0 the same expected utility, then they generate the same expected utility for the seller.

**Proof:** The seller’s expected revenue is

\[
R = \sum_{i=1}^{l} \int_{0}^{1} \cdots \int_{0}^{1} c_i(v) \left( \prod_{i=1}^{l} f_i(v_i) \, dv_i \right)
\]
\[ \sum_{i=1}^{I} \int_{0}^{1} \left[ \int_{0}^{1} \cdots \int_{0}^{1} c_{i}(v) \left( \prod_{j \neq i} f_{j}(v_{j}) \right) dv_{i} \right] f_{i}(v_{i}) dv_{i} = \sum_{i=1}^{I} \int_{0}^{1} \tau_{i}(v_{i}) f_{i}(v_{i}) dv_{i} = \sum_{i=1}^{I} \int_{0}^{1} \left[ \tau_{i}(0) + \bar{p}_{i}(v_{i})v_{i} - \int_{0}^{v_{i}} \bar{p}_{i}(t) dt \right] f_{i}(v_{i}) dv_{i} \]

\[ = \sum_{i=1}^{I} \int_{0}^{1} \left[ \bar{p}_{i}(v_{i})v_{i} - \int_{0}^{v_{i}} \bar{p}_{i}(t) dt \right] f_{i}(v_{i}) dv_{i} + \sum_{i=1}^{I} \tau_{i}(0). \]

Here the fourth equality is from (b) of the last result. \[ \square \]

III. The Optimal Selling Mechanism

A. In view of the last result the sellers problem reduces to choosing a system of assignment probabilities \( p_{i}(\cdot) : [0, 1]^{I} \rightarrow [0, 1] \) and zero value expected costs \( \tau_{i}(0) \) to maximize

\[ R = \sum_{i=1}^{I} \int_{0}^{1} \left[ \bar{p}_{i}(v_{i})v_{i} - \int_{0}^{v_{i}} \bar{p}_{i}(t) dt \right] f_{i}(v_{i}) dv_{i} + \sum_{i=1}^{I} \tau_{i}(0) \]

subject to:

1. \( \sum_{i=1}^{I} p_{i}(v) \leq 1 \) for all \( v \in [0, 1]^{I} \).
2. \( \bar{p}_{i}(\cdot) \) is nondecreasing for all \( i \).
3. \( \tau_{i}(0) \leq 0 \) for all \( i \).

B. There is now a long computation. By bringing the outer integral inside the brackets, changing the order of integration over the region \( \{(v_{i}, t) : 0 \leq v_{i} \leq 1, 0 \leq t \leq v_{i} \} \), then swapping the variables of integration, we obtain

\[ R = \sum_{i=1}^{I} \left[ \int_{0}^{1} \bar{p}_{i}(v_{i})v_{i} f_{i}(v_{i}) dv_{i} - \int_{0}^{1} \left( \int_{0}^{v_{i}} \bar{p}_{i}(t) dt \right) f_{i}(v_{i}) dv_{i} \right] + \sum_{i=1}^{I} \tau_{i}(0) \]

\[ = \sum_{i=1}^{I} \left[ \int_{0}^{1} \bar{p}_{i}(v_{i})v_{i} f_{i}(v_{i}) dv_{i} - \int_{0}^{1} \left( \int_{0}^{1} f_{i}(v_{i}) dv_{i} \right) \bar{p}_{i}(t) dt \right] + \sum_{i=1}^{I} \tau_{i}(0) \]

\[ = \sum_{i=1}^{I} \left[ \int_{0}^{1} \bar{p}_{i}(v_{i})v_{i} f_{i}(v_{i}) dv_{i} - \int_{0}^{1} \left( \int_{v_{i}}^{1} f_{i}(t) dt \right) \bar{p}_{i}(v_{i}) dv_{i} \right] + \sum_{i=1}^{I} \tau_{i}(0) \]
\[ \sum_{i=1}^{I} \int_{0}^{1} p_i(v_i) \left[ v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] f_i(v_i) \, dv_i + \sum_{i=1}^{I} \bar{c}_i(0). \]

C. Recalling that
\[ \bar{p}_i(r_i) = \int_{0}^{1} \cdots \int_{0}^{1} p_i(r_i, v_{-i}) \left( \prod_{j \neq i} f_j(v_j) \, dv_j, \right), \]
we have
\[ R = \int_{0}^{1} \cdots \int_{0}^{1} \left[ \sum_{i=1}^{I} p_i(v_i) \left( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) \right] \left( \prod_{j=1}^{I} f_j(v_j) \, dv_j \right) + \sum_{i=1}^{I} \bar{c}_i(0). \]

1. Since \( \sum_i p_i(v) \leq 1 \) for all \( v \), for any feasible and incentive compatible selling mechanism we must have
\[ R \leq \int_{0}^{1} \cdots \int_{0}^{1} \max \left\{ 0, \max_i \left\{ v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right\} \right\} \left( \prod_{j=1}^{I} f_j(v_j) \, dv_j \right) + \sum_{i=1}^{I} \bar{c}_i(0). \]

D. Designing the Optimal Mechanism

1. Evidently we would like to maximize the sum in square brackets for each \( v \).
2. Therefore we set \( p_i^*(v) = 1 \) if \( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \geq 0 \), \( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} > v_j - \frac{1 - F_j(v_j)}{f_j(v_j)} \)
   for all \( j < i \), and \( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \geq v_j - \frac{1 - F_j(v_j)}{f_j(v_j)} \) for all \( j > i \), and otherwise we set \( p_i^*(v_i) = 0 \).
   a. This allocation rule is inefficient when the agent with the highest virtual valuation does not have the highest valuation. This may viewed as a type of price discrimination.
   b. The outcome may also be inefficient because the seller retains the object in some cases.
   c. In a sense the seller is acting as a monopolist, increasing price by withholding some of the supply.
3. For each \( i \) and \( v_{-i} \) let \( c_i^*(0, v_{-i}) = 0 \). For general \( v \in [0, 1]^I \) let
   \[ c_i^*(v) = c_i^*(0, v_{-i}) + p_i^*(v) v_i - \int_{0}^{v_i} p_i^*(t, v_{-i}) \, dt. \]
   a. Note that \( c_i^*(v) \leq v_i \); if asked to pay, agent \( i \) can still hope to profit.
b. By integrating over the possible \( v_i \) we find that \( \bar{c}_i^*(0) = 0 \) and

\[
\bar{c}_i^*(v_i) = \bar{c}_i^*(0) + \bar{p}_i^*(v_i) v_i - \int_0^{v_i} \bar{p}_i^*(t) \, dt
\]

for all \( v_i \in [0, 1] \).

E. The quantity \( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \) is called (at least by Myerson, in other contexts) a \textit{virtual utility}.

1. It may be thought of as the net revenue generated by giving the object to \( i \) when her value is \( v_i \), which is decreased by a term reflecting the fact that less revenue can be extracted from higher types of agent \( i \).

2. From this point on we assume that for all \( i \), \( v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \) is a strictly increasing function of \( v_i \).

   a. By assuming that the virtual utility is strictly increasing, we insure that the various agents' virtual utilities are distinct with probability one.

   b. Many distributions \( F_i \) give strictly increasing virtual utility. The quantity \( \frac{f_i(v_i)}{1 - F_i(v_i)} \) is called the \textit{hazard rate} because for small \( \Delta > 0 \), \( \Delta \) times the hazard rate is the probability of agent \( i \)'s valuation being in \([v_i, v_i + \Delta)\) conditional on it not having already been found in \([0, v_i)\). If the hazard rate is nondecreasing (as it is for the uniform distribution) then virtual utility is strictly increasing.

3. The main reason for assuming an increasing virtual utility is that it insures that \( \bar{p}_i^* \) is nondecreasing, because \( \bar{p}_i^*(v_i) \) is the probability that \( v_j - \frac{1 - F_i(v_i)}{f_j(v_j)} < v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \) for all \( j \neq i \).

4. Thus, when the virtual utility is nondecreasing, \( (p_i^*(\cdot), c_i^*(\cdot))_{i=1}^I \) is incentive compatible, and thus indeed an optimal auction.
IV. The Symmetric Case

A. We now consider the case where \( F_1 = \cdots = F_I = F \), so that all agents have the same distribution of valuations. Of course \( f_1 = \cdots = f_I = f \).

B. Let \( r \) be the number such that \( r - \frac{1 - F(r)}{f'(r)} = 0 \).

1. We refer to \( r \) as the reserve price.

C. The expected revenue maximizing allocation rule is now simply that the object is allocated to the agent with the highest valuation in excess of the reserve price.

1. If no agent’s value exceeds the reserve price, then (inefficiently) the seller retains the object.

D. We can implement the optimal auction with the payment rule in which the winner pays the maximum of the reserve price and the second highest bid.

1. It is easy to see that bidding truthfully is a dominant strategy for this payment rule.

2. We know that all incentive compatible payment schemes that assign no payments to agents with valuation zero give the same expected revenue, so this must be expected revenue maximizing.

3. Note that when \( I = 1 \), this is a take-it-or-leave-it offer.