I. Introduction.

A. We now begin the theory of games in extensive form.

1. This is more concrete than the strategic form. Most economic models are described in this way. The extensive form is also closer to the “rules” of a game as the term would usually be understood.

2. Notationally these games are much more complex. Eventually we will have symbols for the following types of objects.
   a. States of the game.
   b. Information sets.
   c. Actions, and the set of actions available at each information set.
   d. The assignment of an agent to each information set.
   e. The rule assigning a new state when an allowed action is chosen at a given state.
   f. The payoffs at game-ending states.

B. The transition from normal form games to extensive games is eased if we begin by considering games of perfect information.

1. Examples include chess, checkers, go, and backgammon.

2. The key feature is that the agent choosing the move always knows the state of the game.

C. Though rather simple, this class of games is still interesting in its own right.
1. Historically the first work on extensive games focused on this case.

2. Equilibrium theory for these games is well understood and serves as a useful benchmark for evaluating solution concepts for general extensive games.

II. The Tree.

A. Let $T$ be a finite set of nodes, and let $\prec$ be a strict partial ordering of $T$ denoting precedence.

1. For $t \in T$ let $P(t) = \{x \in T | x \prec t\}$ be the set of predecessors of $t$.

2. **Definition:** We say that the pair $(T, \prec)$ is a tree if:

   (a) there is a node $w \in T$, called the root, that precedes all others.

   (b) for all $t \in T$, $P(t)$ is completely ordered by $\prec$, i.e.

   $$x_1, x_2 \in P(t) \Rightarrow [x_1 \prec x_2 \text{ or } x_1 = x_2 \text{ or } x_2 \prec x_1].$$

   i. In many games, e.g. chess, the same “position” can be reached by different move orders.

   ii. This assumption amounts to choosing to distinguish equivalent states reached by different “histories.”

   iii. This complicates analysis of examples but simplifies the general theory.

B. The immediate predecessor function.

1. For $t \in T - \{w\}$ let $p_1(t) = \max P(t)$ be the immediate predecessor of $t$.

   Note that

   $$p_1(t) \prec t \text{ and } [x \prec t \Rightarrow (x = p_1(t) \text{ or } x \prec p_1(t))].$$

2. There is usually no need to do so, but it is sometimes convenient to set $p_1(w) = w$. With this convention we can define $p_\ell : T \rightarrow T (\ell = 2, 3, \ldots)$ by $p_\ell(t) = p_1(p_{\ell-1}(t))$. Then for all $t \in T$ there exists $\ell$ such that $p_\ell(t) = w$.

3. Conversely, suppose there is a function $p : T \rightarrow T$ with $p(w) = w$ such that
for all \( t \in T \) there is a natural number \( \ell \) with \( p_\ell(t) = w \). Then \((T, \prec)\) is a tree if we define the precedence relation \( \prec \) by declaring that \( x \prec t \) if and only if \( x \neq t \) and there is an \( \ell \) such that \( x = p_\ell(t) \). This is an alternative definition of a tree.

C. Derived notation.

1. \( F(t) = p_1^{-1}(t) = \{ t' | p_1(t') = t \} \) is the set of immediate successors of \( t \).
2. \( Z = \{ z \in T | F(z) = \emptyset \} \) is the set of terminal nodes.
3. \( X = T - Z \) is the set of strategic nodes.
4. \( Y = T - W \) is the set of noninitial nodes.
5. \( Z(x) = \{ z \in Z | x \prec z \} \) is the set of terminal successors of \( x \).

III. Games of Perfect Information.

A. **Definition:** A game of perfect information is a tuple

\[ G = ((T, \prec), (I, \iota), u) \]

where

1. \((T, \prec)\) is a tree,
2. \( I \) is a finite set of agents,
3. \( \iota : X \to I \) is a function, and
4. \( u = (u_i)_{i \in I} \) is a vector of utility function \( u_i : Z \to \mathbb{R} \). (Alternatively, we may regard \( u \) as an element of \( \mathbb{R}^{Z \times I} \).)

B. Informally, the rules of \( G \) are that \( \iota(w) \) chooses \( t_1 \in F(w) \), \( \iota(t_1) \) chooses \( t_2 \in F(t_1) \), and so on until a terminal node is reached.

C. We could allow chance nodes were the next move is chosen randomly, for example by rolling dice. We have not done so only because it does not really add any ideas to the discussion below.
IV. The Derived Normal Form

A. Let $X_i = i^{-1}(i) = \{x \in X | \iota(x) = i\}$.

B. **Definition:** A pure strategy for $i$ is a function $s_i : X_i \to T$ with $s_i(x) \in F(x)$ for all $x \in X_i$. Let $S_i$ be the set of pure strategies for $i$, and let $S = \Pi_{i \in I} S_i$.

**Important Remark:** A strategy profile $s$ may be regarded as a function $s : X \to T$ satisfying $s(x) \in F(x)$ for all $x \in X$. Conversely, any such function may be regarded as a strategy profile.

C. **Definition:** If $s$ is a strategy profile, define $t_0(s) = w$, $t_1(s) = s(w)$, $t_2(s) = s(t_1(s))$, and so on. The terminal node determined by $s$ is $z(s) = t_{\ell(s)}(s)$ where $\ell(s)$ is the integer with $t_{\ell(s)}(s) \in Z$.

D. **Definition:** The normal form of $G$ is

$$N(G) = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$$

where the $S_i$ are those defined above and, abusing notation, $u_i(s) = u_i(z(s))$.

V. Subgames and Subgame Perfect Equilibrium

A. Introduction

1. Common knowledge of rationality implies that rationality should be expected in all contingencies.

2. This motivates interest in subgames, and notions of equilibrium based on them.

B. For $x \in X$ let

$$T^x = \{x\} \cup \{T \mid x \prec t\},$$

$$\prec^x = \prec \cap (T^x \times T^x),$$

$$\iota^x = \iota|_{X \cap T^x},$$

$$u^x_i = u_i|_{T^x \cap Z}.$$

C. **Definition:** The subgame beginning at $x$ is

$$G^x = ((T^x, \prec^x), (I, \iota^x), u^x)$$
1. If $s_i \in S_i$ is a normal form strategy for $i$ in $G$, let $s_i^x = s_i|_{X_i \cap T^x}$.

2. If $s \in S$ let $s^x = (s_i^x)_{i \in I}$.

**D. Definition:** A *pure strategy subgame perfect equilibrium* for $G$ is a normal form strategy profile $s$ such that, for all $x$, $s^x$ is a Nash equilibrium of $N(G^x)$.

**Theorem:** Every game of perfect information has at least one pure strategy subgame perfect equilibrium.

**Proof:** Let $x$ be a strategic node such that $G^x$ has a p.s.s.p.e. for all $x' \in X$ with $x \prec x'$. Let $F(x) = \{t_1, \ldots, t_J\}$, and for those $t_j$ in $X$ let $s_j^t$ be a p.s.s.p.e. of $G_j^t$. Define $z(t_j)$ to be $t_j$ if $t_j \in Z$, and otherwise let $z(t_j) = s_j^t$. Choose

$$s_i(x)(x) \in \text{argmax}_{t \in F(x)} u_i(z(t_j)).$$

We have defined all components of a strategy profile $s : T^x \cap X \rightarrow T^x$. It is obvious that $s$ is a Nash equilibrium of $G^x$, so $s$ is a p.s.s.p.e. □

**E. Definition:** A game $G$ is *without indifference between outcomes* if there is a space of outcomes $\Omega$ and maps $\omega : Z \rightarrow \Omega$ and $v_i : \Omega \rightarrow \mathbb{R}$ such that $u_i = v_i \circ \omega$ and $v_i(\omega) \neq v_i(\omega')$ whenever $\omega \neq \omega'$.

1. In chess, for example, there are many terminal nodes, but $\Omega = \{\text{win, loss, draw}\}$.

**Proposition 1:** (Zermelo-Kuhn) If $G$ is without indifference between outcomes, for all $x \in X$ there is $\omega^x \in \Omega$ such that $\omega(z(s^x)) = \omega^x$ for all p.s.s.p.e. $s^x$ of $G^x$.

**Proof:** As in the proof above, let $F(x) = \{t_1, \ldots, t_J\}$ and suppose that each $t_j$ is either terminal, in which case we set $\omega_j^t = \omega(t_j)$, or has $\omega_j^t \in \Omega$ with $\omega(z(s_j^t)) = \omega_j^t$ for all p.s.s.p.e. $s_j^t$. If $s^x$ is a p.s.s.p.e. of $G^x$ then

$$s^x(x) = \text{argmax}_{t \in F(x)} v_i(x)(\omega^t) \text{ and }$$
$$\omega(z(x^x)) = \omega^x = \text{argmax}_{t \in F(x)} v_i(x)(\omega^t).$$
The claim now follows by backward induction. ■

F. **Definition:** A game \( G \) is **without indifference between terminal nodes** if \( u_i(z) \neq u_i(z') \) whenever \( z \neq z' \).

**Proposition 2:** If \( G \) is without indifference between terminal nodes, then there is a unique p.s.s.p.e.

**Proof:** Let \( \Omega = Z \) in Proposition 1. ■

VI. Problems with Nash Equilibrium

A. We present examples of unreasonable Nash equilibria, then a general concept to deal with them. We then consider more examples to see how it works.

B. The incredible threat

1. **Incredible Threat:** Player 1 first choose between \( U \) and \( D \). If she chooses \( U \) the game ends with the payoff vector \((1, 2)\). If she chooses \( D \) then player 2 chooses between \( L \) with payoff vector \((2, 1)\) and \( R \) with payoff vector \((0, 0)\).

2. The normal form of this game is

\[
\begin{array}{c|cc}
   & L & R \\
\hline
U & (1, 2) & (1, 2) \\
D & (2, 1) & (0, 0) \\
\end{array}
\]

3. \((D, L)\) and \((U, R)\) are both equilibria, and in fact \((U, \alpha L + (1 - \alpha)R)\) is an equilibrium for all \( \alpha \leq \frac{1}{2} \). Only \((D, L)\) is "reasonable" in the sense that player 1 should expect player 2 to be rational in any contingency.

C. Behavior inconsistent with any beliefs.

1. Player 1 choose between \( U, C, \) and \( D \), which ends the game with payoff vector \((1, 4)\). After \( U \) or \( C \) player 2 chooses between \( \ell, m, \) and \( r \).
2. The corresponding normal form is

\[
\begin{pmatrix}
1 \backslash 2 & \ell & m & r \\
U & (2, 3) & (0, 1) & (2, 0) \\
C & (2, 0) & (0, 1) & (2, 3) \\
D & (1, 4) & (1, 4) & (1, 4)
\end{pmatrix}
\]

3. There are two types of Nash equilibria.
   a. \((\alpha U + (1 - \alpha)C, \beta \ell + (1 - \beta)r)\) where \(\beta = 0\) if \(\alpha < \frac{1}{2}\), \(\beta\) can be any probability if \(\alpha = \frac{1}{2}\), and \(\beta = 1\) if \(\alpha > \frac{1}{2}\).
   b. \((D, \gamma \ell + \delta m + \varepsilon r)\) such that \(\gamma + \delta + \varepsilon = 1\) and \(\delta \geq \frac{1}{2}\).

4. The second class of equilibria is unreasonable since agent 2’s behavior is not optimal for any belief about the relative likelihood of \(U\) and \(C\).

D. Equilibria using dominated strategies

1. Consider the normal form

\[
\begin{pmatrix}
1 \backslash 2 & L & R \\
U & (1, 1) & (0, 0) \\
D & (0, 0) & (0, 0)
\end{pmatrix}
\]

2. \((U, L)\) and \((D, R)\) are both equilibria, but \((D, R)\) uses weakly dominated strategies, and seems peculiar.

3. One should retain an open mind on this, since one has to allow \((D, R)\) if one wants the equilibrium correspondence to have a closed graph.

VII. A Model of Mistakes with Small but Positive Probability

A. The intuitive idea – each agent is forced to assign a small but positive probability to each of his pure strategies.

1. One cannot have equilibrium strategy profiles with incredible threats, since the situation in which the threat is to be carried out occurs with positive probability.
2. Bayesian posterior beliefs are defined everywhere in the game tree.
3. Weakly dominated strategies are distinctly inferior.

B. The formal development

1. Let \( N = (S_1, \ldots, S_n; u_1, \ldots, u_n) \) be a normal form game.

2. **Definition:** A **tremble** for \( N \) is a vector of functions \( \varepsilon = (\varepsilon_i)_{i \in I} \) where each \( \varepsilon_i : S_i \to (0, 1] \) satisfies \( \sum_{s_i} \varepsilon_i(s_i) \leq 1 \).

3. **Definition:** Given an agent \( i \), a strategy profile \( \sigma \), and a tremble \( \varepsilon \), the \( \varepsilon \)-**best response set** for \( i \) is

\[
BR^\varepsilon_i(\sigma) = \{ \sigma'_i | \sigma'_i(s_i) \geq \varepsilon_i(s_i) \text{ with strict inequality only if } s_i \in BR_i(\sigma), \text{ all } s_i \in S_i \}\]

4. **Definition:** The \( \varepsilon \)-**best response correspondence** is \( BR^\varepsilon : \Sigma \to \Sigma \) defined by

\[
BR^\varepsilon(\sigma) = \Pi_{i \in I} BR^\varepsilon_i(\sigma).
\]

An \( \varepsilon \)-**perfect equilibrium** is a fixed point of \( BR^\varepsilon \).

**Lemma:** For any tremble \( \varepsilon \) there is an \( \varepsilon \)-perfect equilibrium.

**Proof:** We leave it as an exercise to verify that \( BR^\varepsilon \) is an upper semicontinuous convex valued correspondence. After this fact has been verified the result follows from Kakutani’s fixed point theorem.

5. **Definition:** A strategy profile \( \sigma^* \in \Sigma \) is a **perfect equilibrium** if there is a sequence of trembles \( \{\varepsilon^r\} \) with

\[
(*) \quad \lim_{r \to \infty} (\max_{i \in I, s_i \in S_i} \varepsilon^r_i(s_i)) = 0
\]

and a sequence \( \{\sigma^r\} \), where each \( \sigma^r \) is an \( \varepsilon^r \)-perfect equilibrium and \( \sigma^r \to \sigma^* \).

**Theorem 1:** There are perfect equilibria.

**Proof:** Let \( \{\varepsilon^r\} \) be a sequence of trembles satisfying \((*)\), and for each \( r \) let \( \sigma^r \) be an \( \varepsilon^r \)-perfect equilibrium. Since \( \Sigma \) is compact, the sequence \( \{\sigma^r\} \) has a convergent subsequence.
Theorem 2: Perfect equilibria are Nash equilibria.

Proof: Let \( \{ \varepsilon^r \} \) and \( \{ \sigma^r \} \) be as in the definition of perfect equilibrium, with \( \sigma^r \to \sigma^* \).
Consider \( i \in I \) and \( \sigma_i \in \Sigma_i \). If \( s_i \) is not a best response for \( i \) to \( \sigma^* \), then it is not a best response to \( \sigma^r \) for large \( r \), since \( u_i \) is continuous, so that \( \sigma^r_i(s_i) = \varepsilon^r_i(s_i) \) and \( \sigma^*_i(s_i) = 0 \). Thus \( \sigma^*_i \in BR_i(\sigma^*) \).

VIII. The Examples Revisited

[Reanalyze the three examples of the introduction.]

IX. A Key Example (Myerson)

A. Consider the following normal form game.

\[
\begin{array}{c|ccc}
1 & L & R & F \\
\hline
U & (1, 1) & (0, 0) & (-9, -9) \\
D & (0, 0) & (0, 0) & (-7, -7) \\
W & (-9, -9) & (-7, -7) & (-7, -7) \\
\end{array}
\]

1. Without \( W \) and \( F \), \((U, L)\) is the only perfect equilibrium.
2. With \( W \) and \( F \), \((D, R)\) is a perfect equilibrium, as we can see by considering the tremble \( \varepsilon(U) = \varepsilon(D) = \varepsilon(W) = \varepsilon(L) = \varepsilon(R) = \varepsilon(F) = \alpha \).

B. This raises an important point about game theoretic modelling.

1. When we write down a game, typically we leave out some strongly dominated strategies.
2. Intuitively one would expect the presence or absence of strongly dominated strategies to have no effect when it is common knowledge that all players are rational.
3. For this reason a desirable property of solution concepts is that they are invariant with respect to elimination of strongly dominated strategies, and by induction, the repeated elimination of such strategies.
X. The Idea

A. Some “mistakes” are worse than others, and properness is derived from an assumption that more costly mistakes are “infinitely less likely” than less costly mistakes.

B. It is easy to think of counterexamples to this in human behavior. Poker praxis is particularly rich with examples.

C. A belief in the usefulness or interest of properness does not require a belief in the relative improbability of more costly mistakes.
   1. One may simply like solution concepts to be restrictive.
   2. The properties of proper equilibria may be used to prove theorems about other solution concepts.

XI. The Formal Execution

A. Let \( N = (S_1, \ldots, S_n; u_1, \ldots, u_n) \) be a normal form game.

B. **Definition:** \( \sigma_i \in \Delta(S_i) \) is *totally mixed* if \( \sigma_i(s_i) > 0 \) for all \( s_i \in S_i \), and \( \sigma \in \Sigma \) is *totally mixed* if each of its components is totally mixed.

C. **Definition:** For any \( \eta > 0 \), an \( \eta \)-proper equilibrium is a totally mixed \( \sigma \in \Sigma \) such that for all \( i \in I \) and \( r_i, s_i \in S_i \),

\[
    u_i(s_i, \sigma_{-i}) < u_i(r_i, \sigma_{-i}) \quad \text{implies} \quad \sigma_i(s_i) \leq \eta \sigma_i(r_i).
\]

**Remark:** An \( \eta \)-proper equilibrium is an \( \eta \)-perfect\(^*\) equilibrium.

D. **Definition:** \( \sigma^* \) is a proper equilibrium if there are sequences \( \{\eta^r\}_{r=1,2,\ldots} \subset (0, 1) \) and \( \{\sigma^r\}_{r=1,2,\ldots} \subset \Sigma \) such that \( \eta^r \to 0 \), each \( \sigma^r \) is an \( \eta^r \)-proper equilibrium, and \( \sigma^r \to \sigma^* \).

**Proposition:** The set of proper equilibria is a subset of the set of perfect equilibria, and the containment may be strict.
Theorem: The set of proper equilibria is nonempty.

Proof: It suffices to prove the existence of an \( \eta \)-proper equilibrium for any \( \eta \in (0, 1) \), since one can then take a convergent subsequence of a sequence \( \{\sigma^r\} \) where each \( \sigma^r \) is an \( \eta^r \)-proper equilibria and \( \eta^r \to 0 \).

Fix \( \eta \in (0, 1) \). Let \( Q = \max_{i \in I} \#(S_i) \). For each \( i \in I \) define \( BR_i^{P(\eta)} : \Sigma \to \Delta(S_i) \) by

\[
BR_i^{P(\eta)}(\sigma) = \{\sigma'_i / \sigma'_i \in \Delta(S_i) | \sigma'_i(s_i) \geq \frac{\eta Q}{Q} \text{ for all } s_i \in S_i, \text{ and for all } r_i, s_i \in S_i, u_i(r_i, \sigma_{-i}) < u_i(s_i, \sigma_{-i}) \text{ implies } \sigma'_i(r_i) \leq \eta \sigma'_i(s_i) \}.
\]

Lemma 1: \( BR_i^{P(\eta)}(\sigma) \neq \emptyset \).

Proof: If \( S_i = \{s_i^1, \ldots, s_i^{G_i}\} \) where \( u_i(s_i^{q-1}, \sigma_{-i}) \leq u_i(s_i^q, \sigma_{-i}) \), then \( \sigma'_i \) given by

\[
\sigma'_i(s_i^g) = \frac{\eta Q - g}{Q}, \quad g = 1, \ldots, G_i - 1, \quad \text{and} \\
\sigma'_i(s_i^{G_i}) = 1 - \frac{\eta Q - G_i + 1 - \eta Q}{(1 - \eta)Q}
\]

is in \( BR_i^{P(\eta)}(\sigma) \). \( \blacksquare \)

Lemma 2: \( BR_i^{P(\eta)} \) is upper semicontinuous and convex valued.

Proof: If \( \sigma_i^m \in BR_i^{P(\eta)}(\sigma^m), m = 1, 2, \ldots, \sigma^m \to \sigma, \) and \( \sigma_i^m \to \sigma'_i \), then \( \sigma'_i(r_i) > \eta \sigma'_i(s_i) \) implies that \( \sigma'_i^m(r_i) > \eta \sigma'_i^m(s_i) \) for large \( m \). Since \( \sigma_i^m \in BR_i^{P(\eta)}(\sigma^m) \), it follows that \( u_i(r_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}) \) for large \( m \), so that \( u_i(r_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}) \).

Thus \( \sigma'_i \in BR_i^{P(\eta)}(\sigma) \).

\( BR_i^{P(\eta)}(\sigma) \) is convex since it is defined by linear inequalities. \( \blacksquare \)
Defining $BR^P(\eta) : \Sigma \to \Sigma$ by $BR^P(\eta)(\sigma) = \Pi_{i \in I} BR^P_i(\eta)(\sigma)$, Lemmas 1 and 2 imply that the hypotheses of the Kakutani fixed point theorem are satisfied. Thus $BR^P(\eta)$ has a fixed point which must be an $\eta$-proper equilibrium. 

E. Every proper equilibrium is a perfect equilibrium. You should think about the idea of the proof, which is fairly simple, even if the details are a bit messy.

XII. Imperfections of Perfection and Properness

A. Adding strictly dominated strategies can make equilibria perfect that were not perfect before.

$$
\begin{array}{c|ccc}
1 & L & R & M \\
\hline
U & (1, 1) & (0, 0) & (1, -1) \\
D & (0, 0) & (0, 0) & (2, -1) \\
C & (-1, 1) & (-1, 2) & (-1, -1) \\
\end{array}
$$

1. $(\alpha U + (1-3\alpha)D + 2\alpha C, \alpha L + (1-3\alpha)R + 2\alpha M)$ is an $\varepsilon$-perfect equilibrium for an appropriate tremble $\varepsilon$.

B. The Kohlberg Example

1. Draw the extensive form.

2. The normal form is

$$
\begin{array}{c|cc}
1 & \ell & r \\
\hline
U & (3, 3) & (0, 0) \\
M & (0, 0) & (1, 1) \\
D & (2, 2) & (2, 2) \\
\end{array}
$$

3. Here $M$ is a dominated action, but adding it to the game makes $(D, r)$ a perfect equilibrium whereas it was not perfect before.

XIII. General Extensive Form Games

A. Now let $(T, \prec)$ be, in effect, a finite collection of trees:

1. The nodes in $T$ are strictly partially ordered by the relation $\prec$, which is called precedence.
a. \( \prec \) is asymmetric: it is never the case that \( t \prec t \).

b. \( \prec \) is transitive: if \( x \prec y \prec z \) then \( x \prec z \).

2. For \( t \in T \), \( P(t) = \{ x \in T | x \prec t \} \) is the set of predecessors of \( t \).

3. The pair \((T, \prec)\) is an arborescence if, for all \( t \), \( P(t) \) is completely ordered by \( \prec \): if \( x, x' \in P(t) \) and \( x \neq x' \), then \( x \prec x' \) or \( x' \prec x \).

B. In an arborescence there are a number of subsets of \( T \) that have particular roles.

1. The set of initial nodes is \( W = \{ w \in T | P(w) = \emptyset \} \). These are the possible starting points of the game.

2. The set of nonterminal or strategic nodes is \( X = \{ x \in T | P^{-1}(x) \neq \emptyset \} \). These are points at which a decision is made.

3. Let \( Y = T - W = \{ y \in T | P(y) \neq \emptyset \} \) be the set of noninitial nodes. We think of them as nodes that are possible consequences of choices.

4. The set of terminal nodes is \( Z = \{ z \in T | P^{-1}(z) = \emptyset \} \). These are nodes at which the game is over.

5. The following auxiliary notation will be useful.

   a. Let \( p : Y \to X \) be defined by \( p(y) = \max P(y) \). That is, \( p(y) \) is the unique node such that \( P(y) = \{ p(y) \} \cup P(p(y)) \). We call \( p(y) \) the immediate predecessor of \( y \).

   b. For \( t \in T \) let \( p_0(t) = t \). For \( \ell \geq 1 \) and \( y \in Y \) such that \( p_{\ell-1}(y) \in Y \) let \( p_\ell(y) = p(p_{\ell-1}(y)) \). Let \( \ell(t) \) be the integer such that \( p_{\ell(t)}(t) \in W \). We call \( p_\ell(t) \) the \( \ell \)th predecessor of \( t \) and \( \ell(t) \) the level of \( t \).

   c. For \( x \in X \) let \( F(x) = p^{-1}(x) \) be the set of immediate successors of \( x \).

C. Information and choices.

1. The information partition is \( H \), a partition of \( X \).

   a. This means that \( H \) is a collection of nonempty pairwise disjoint subsets of \( X \) whose union is \( X \).

   b. Elements of \( H \) are called information sets.
c. The idea is that when a node in \( h \in H \) occurs during a play of the game, the agent who chooses an action knows that some node in \( h \) has occurred, but he does not know which one.

d. For \( x \in X \), let \( H(x) \) be the information set that contains \( x \).

2. There is a finite set \( A \) of actions and a map \( \alpha : Y \to A \).

   a. We call \( \alpha(y) \) the action chosen immediately prior to \( y \).

   b. It must be the case that the choice of an action at a strategic node determines an immediate successor uniquely.

      i. Therefore we assume that \( \alpha(y) = \alpha(y') \) implies that \( p(y) \neq p(y') \).

      ii. Let \( A(x) = \alpha(F(x)) \) be the set of actions available at \( x \).

   c. An agent should not need to know which node in an information set of his has occurred in order to know which actions are available, so we assume that \( A(x) = A(x') \) for all \( h \in H \) and \( x, x' \in h \).

      i. Let \( A(h) = A(x), x \in h, \) be the set of actions available at \( h \).

      ii. We will assume that \( \{ A(h) \mid h \in H \} \) is a partition of \( A \).

D. Agents.

   1. Let \( I \) be a finite set of agents or players.

      a. Typically we assume that \( I = \{1, \ldots, n\} \) for some integer \( n \).

   2. There is a function \( \iota : H \to I \) that specifies the agent who makes a choice at each information set.

E. A tuple \( ((T, \prec), H, (A, \alpha), (I, \iota)) \) with the properties detailed above is called an extensive form.

F. We now specify objects defined in terms of real numbers.

   1. The initial assessment is a probability measure \( \rho \in \Delta(W) \).

      a. Often one assumes that \( \rho \) is interior: \( \rho(w) > 0 \) for all \( w \in W \).

      b. For the most part the theory extends without difficulty to a specification in which different agents have different initial assessments.
i. This creates a certain amount of notational complication.

ii. An explanation of an economic phenomenon based on divergent initial beliefs seems ad hoc, so the assumption of a common prior (sometimes called the Harsanyi doctrine or consistency in the sense of Harsanyi) forces a certain discipline on our modelling methods.

c. This formulation puts all of the randomness in the selection of the initial node, instead of having chance nodes. This is conceptually and notationally simpler, but complicates the description of particular examples.

2. The payoff or utility is \( u \in \mathbb{R}^{Z \times I} \).
   a. Here \( u_i \in \mathbb{R}^Z \) is the utility function for agent \( i \).
   b. Similarly \( u_z \in \mathbb{R}^I \) is the payoff vector at the terminal node \( z \).

3. An extensive form game is a tuple

\[
G = ((T, \prec), H, (A, \alpha), (I, \iota), \rho, u)
\]

with the properties detailed above.

G. It is natural to assume that agents remember everything that has happened to them.

1. More precisely, any two nodes \( x \) and \( x' \) in an information set must have the same action history for the agent who chooses there.

2. We introduce the following notation:
   a. For \( i \in I \), let \( H_i = \iota^{-1}(i) \) be the set of information sets at which agent \( i \) chooses, and let \( A_i = \bigcup_{h \in H_i} A(h) \) be the set of all actions that agent \( i \) might ever choose.
   b. For \( t \in T \), let \( C(t) = \emptyset \) if \( t \in W \), and if \( t \in W \) let

\[
C(t) = \{ \alpha(t) \} \cup \{ \alpha(y) : y \in P(t) \setminus W \}.
\]

This is the history of action choices prior to \( t \).

**Perfect Recall:** For all \( h \in H \) and \( x, x' \in h \), \( C(x) \cap A_i(h) = C(x') \cap A_i(h) \).
3. Since the sets $A(h)$ partition $A$, this means that the set of information sets of $\iota(h)$ visited prior to $x$ is the same as the set of information sets of $\iota(h)$ visited prior to $x'$, and at each of those information sets the action on the path to $x$ is the same as the action on the path to $x'$.

XIV. Pure Strategies and the Normal Form

A. A pure strategy for agent $i$ is a function $s_i: H_i \rightarrow A$ with $s_i(h) \in A(h)$ for all $h \in H_i$.

1. Intuitively a pure strategy for $i$ is a plan of action specifying a decision in every situation in which agent $i$ must make a choice.

2. There is a certain amount of redundancy here insofar as a pure strategy specifies choices at information sets that are precluded by the strategy itself.

B. A pure strategy profile is a vector $s = (s_1, \ldots, s_n)$ where each $s_i$ is a pure strategy for agent $i$.

1. We may also think of a pure strategy profile as a function $s: H \rightarrow A$ with $s(h) \in A$ for all $h$.

2. Given a pure strategy profile $s$ and a node $t$ there is a unique terminal node $z(t, s)$ that results when strategy $s$ is followed beginning at $t$.

   a. For $x \in X$ and $a \in A(x)$ let $c(x, a)$ be the unique element of $F(x)$ with $\alpha(c(x, a)) = a$. We call $c(x, a)$ the consequence of choosing $a$ at $x$.

   b. Formally the function $z(\cdot, s): T \rightarrow Z$ is defined inductively by requiring that $z(z, s) = z$ for all $z \in Z$ and that $z(x, s) = z(c(x, s(H(x))), s)$ for all $x \in X$.

C. The expected payoff for agent $i$ when strategy profile $s$ is played is

$$u_i(s) = \sum_{w \in W} u_i(z(w, s))\rho(w).$$

D. The normal form of the extensive form game $G = ((T, \prec), H, (A, \alpha), (I, \iota), \rho, u)$ is $N(G) = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ where each $S_i$ is the set of pure strategies for agent $i$ and $u_i: S_1 \times \cdots \times S_n \rightarrow \mathbb{R}$ is the expected utility function defined above.
XV. The Agent Normal Form

A. When Selten’s notion of perfect equilibrium is applied to the normal form, it does not do an adequate job of capturing the notion of conditional rationality.

1. If one of your information sets cannot occur unless you choose some action at an earlier information set that is inferior (relative to some given mixed strategy profile), then your \( \epsilon \)-best responses do not need to prescribe rational, or approximately rational, behavior at the lower information set.

2. The reason is that an \( \epsilon \)-best response will assign only the tremble probability to any normal form pure strategy that allows the lower information set to occur, so that your behavior at the lower information set is dictated by the tremble, not rationality.

B. It turns out that perfection works fairly well when applied to the agent normal form of an extensive game satisfying perfect recall.

1. The idea of the agent normal form is that each information set is treated as a separate (normal form) agent.

2. Formally the agent normal form is

\[
AN(G) = \left( (A(h))_{h \in H}, (u_h)_{h \in H} \right)
\]

where

\[
u_h = u_{i(h)} \quad \text{for all} \quad h \in H.
\]

   a. To understand this recall that a pure strategy profile \( s \) for the normal form is a vector of strategies \( s_i \), where each \( s_i \) is a function specifying an action at each information set where \( i \) chooses.

   b. Thus a pure strategy profile of the normal form amounts to a specification of an action at each information set in the tree, but this is precisely what a pure strategy profile for the agent normal form is. The normal form and the agent normal form have the same spaces of pure strategy profiles.

C. Several philosophical questions arise in connection with the agent normal form.

1. Does the agent normal form (or the extensive form) contain important information that is lost, or obscured, in passing to the normal form?
a. Some game theorists argue that, in principle, the answer should be no, since the two representations represent the same strategic interaction.
b. We will see that certain solution concepts, notably perfection, perform better in the agent normal form, but it can be argued that this is symptomatic of problems with these solution concepts.

2. Is an equilibrium of the agent normal form a “true” equilibrium?
a. That is, is it possible to have a Nash equilibrium of the agent normal form in which some \( i \in I \) can increase his expected payoff by modifying his behavior at several information sets simultaneously?
b. Roughly, we can fail to have a true equilibrium in two ways.

i. At an information set “high up” in the game tree you may assign probability zero to some action because you are afraid that later you will do something bad at an information set that can result from the action. Any behavior at the lower information set is then utility maximizing for the game as a whole because the lower information set never occurs. Perfect equilibria of the agent normal form will not have this problem.

ii. In games without perfect recall, an equilibrium may fail to take advantage of the possibility of correlating behavior across information sets. This sort of difficulty cannot arise in a game of perfect recall because strategic probabilities at an information set “high up” in the game tree cannot affect the conditional probabilities of the nodes in a lower information set that is controlled by the same player.

3. Is an agent \( i \in I \) who is restricted to agent normal form mixed strategies “crippled,” insofar as there might be preferable normal form mixed strategies?
a. As we will see, there may be normal form mixed strategies that are not
derived in any obvious way from agent normal form mixed strategies.

b. A theorem of Kuhn, which we now describe in detail (but do not prove), shows that for any normal form mixed strategy there is an agent normal form mixed strategy that is equivalent in a strong sense.

XVI. Comparison of Mixed Strategies for the Two Normal Forms

A. Notation.

1. Our notation for mixed strategies in “the” normal form is as before.
   a. A mixed strategy for agent $i$ is a probability measure $\sigma_i \in \Delta(S_i)$. We write $\sigma_i(s_i)$ to denote the probability this measure assigns to $s_i$, and for $T_i \subset S_i$ we let $\sigma_i(T_i) = \sum_{s_i \in T_i} \sigma_i(s_i)$.
   b. A mixed strategy profile is a vector

   \[ \sigma = (\sigma_i)_{i \in I} \in \Sigma = \prod_{i \in I} \Delta(S_i). \]

2. Our notation for mixed strategies in the agent normal form is parallel to this.
   a. A mixed strategy for information set $h$ is a probability measure $\pi_h \in \Delta(A_h)$. We write $\pi_h(a_h)$ to denote the probability this measure assigns to $a_h$, and for $B_h \subset A_h$ we let $\pi_h(B_h) = \sum_{a_h \in B_h} \pi_h(a_h)$.
   b. A behavior strategy is a vector

   \[ \pi = (\pi_h)_{h \in H} \in \Pi = \prod_{h \in H} \Delta(A_h). \]
   c. Grouping together those information sets at which a single agent chooses, we obtain a second notion of a mixed strategy for an agent.

   i. A behavior strategy for agent $i$ is a vector

   \[ \pi = (\pi_h)_{h \in H_i} \in \Pi_i = \prod_{h \in H_i} \Delta(A_h). \]

   ii. We see that a behavior strategy may be viewed either as a profile of mixed strategies, one for each information set, or as a profile of
behavior strategies, one for each agent. We will sometimes regard 
\( \Pi \) as the cartesian product of the sets \( \Pi_i \).

**B. The maps from behavior strategies to normal form mixed strategies.**

1. A behavior strategy for agent \( i \) induces a mixed strategy for \( i \) in which the probability of one of agent \( i \)'s pure strategies, say \( s_i \), is the product of the probabilities assigned to the actions \( s_i(h) \) specified by \( s_i \) at the various informations sets in \( H_i \).

2. Formally we define a function \( \lambda_i : \Pi_i \to \Delta(S_i) \) by the formula
   \[
   \lambda_i(\pi_i)(s_i) = \prod_{h \in H_i} \pi_h(s_i(h)).
   \]
   (Here \( \lambda_i(\pi_i)(s_i) \) is the probability the mixed strategy \( \lambda_i(\pi_i) \) assigns to the pure strategy \( s_i \).)

3. Let \( \lambda : \Pi \to \Sigma \) be the function given by the formula
   \[
   \lambda((\pi_i)_{i \in I}) = (\lambda_i(\pi_i))_{i \in I}.
   \]

**XVII. Kuhn’s Theorem**

**A. Posing the problem.**

1. In principle an agent can choose to implement any one of his pure strategies.

2. By randomizing before the game begins he can therefore implement any of his mixed strategies.

3. In general, however, the image of \( \lambda_i \) is a proper subset of \( \Delta(S_i) \), so one has to consider whether an agent loses any significant “strategic flexibility” when he is restricted to his set of behavior strategies.

4. Kuhn’s theorem says that nothing of any importance is lost by restricting attention to behavior strategies. Specifically, it asserts that for each \( i \) and \( \sigma_i \in \Delta(S_i) \) there is some \( \pi^i \in \Pi_i \) that is “realization equivalent” in the sense that, for all \( \tau_{-i} \in \Sigma_{-i} \), \((\sigma_i, \tau_{-i})\) and \((\lambda_i(\pi^i), \tau_{-i})\) induce the same probability distribution on terminal nodes.
B. Kuhn’s theorem requires the assumption of perfect recall.

1. Consider a game without perfect recall in which agent 1 chooses from a set of actions $A_{h_1}$ and then, “without remembering what he did,” chooses from a second set of actions $A_{h_2}$.

2. The set of mixed strategies allows any distribution over $A_{h_1} \times A_{h_2}$ whatsoever.

3. The set of behavior strategies allows only distributions over $A_{h_1} \times A_{h_2}$ in which the choice of $a_1 \in A_{h_1}$ is uncorrelated with the choice of $a_2 \in A_{h_2}$, i.e. $a_1$ and $a_2$ are statistically independent.

C. Motivation and statement of Kuhn’s theorem.

1. Two mixed strategies $\sigma_i, \sigma'_i \in \Sigma_i$ are realization equivalent if

$$\sigma_i(\{s_i \in S_i|z(w, (s_i, t_{-i})) = z\}) = \sigma'_i(\{s_i \in S_i|z(w, (s_i, t_{-i})) = z\})$$

for all $t_{-i} \in S_{-i}$, all initial nodes $w$, and all terminal nodes $z$. Equivalently, for any $\sigma_{-i} \in \Sigma_{-i}$, $(\sigma_i, \sigma_{-i})$ and $(\sigma'_i, \sigma_{-i})$ induce the same distribution on $Z$.

2. Suppose we fix an agent $i$ and a mixed strategy $\sigma_i$ and ask what properties a behavior strategy $\pi^i$ must have in order for $\lambda_i(\pi^i)$ and $\sigma_i$ to be realization equivalent.

3. Consider an information set $h \in H_i$ and a node $x \in h$. Let $J_h(x) \subset H_i$ be the set of information sets $j \in H_i$ for agent $i$ containing nodes that preceed $x$, and for each $j \in J_h(x)$ let $a_h(j, x) = \alpha(y)$ where $y$ is the node preceeding $x$ (or equal to $x$) with $p(y) \in j$.

   a. The assumption of perfect recall says that if $x'$ is another node in $h$ then $J_h(x') = J_h(x)$ and $a_h(j, x') = a_h(j, x)$ for all $j \in J_h(x')$.

   b. Let $J_h$ be the set of information sets for $i$ that preceed $h$, and for $j \in J_h$ let $a_h(j)$ be the action chosen at $j$ on the way to $h$. (Formally $J_h$ is $J_h(x)$ and $a_h(j)$ is $a_h(j, x)$ for any $x \in h$.)

   c. We say that $s_i \in S_i$ allows $h$ if $s_i(j) = a_h(j)$ for all $j \in J_h$. 
4. One possibility is that \( \sigma_i(\{s_i|s_i \text{ allows } h\}) = 0 \).
   a. It must of course be the case that \( \lambda_i(\pi^i) \) also assigns no probability to any pure strategy that allows \( h \).
      i. This means that there must be some \( j \in J_h \) such that \( \pi_h(a_h(j)) = 0 \).
   b. On the other hand \( \pi_h \) may be chosen arbitrarily: since agent \( i \) never allows information set \( h \) to occur, the distribution over actions in \( A_h \) has no effect on the induced distribution over terminal nodes.

5. Now suppose that \( \sigma_i(\{s_i|s_i \text{ allows } h\}) > 0 \).
   a. Naively one might conjecture that we should have \( \pi_h(a) = \sigma_i(\{s_i|s_i(h) = a\}) \) for all \( a \in A_h \).
      i. This ignores the fact that \( s_i(h) \) is not relevant unless \( s_i \) allows \( h \).
   b. The correct formula is that \( \pi_h(a) \) is the probability of \( a \) being chosen conditional on \( h \) being allowed:
      \[
      \pi_h(a) = \frac{\sigma_i(\{s_i|s_i \text{ allows } h \text{ and } s_i(h) = a\})}{\sigma_i(\{s_i|s_i \text{ allows } h\})} \tag{*}
      \]
   c. Note the effect of perfect recall: the set of pure strategies for \( i \) that allow one node in \( h \) is the same as the set of pure strategies for \( i \) that allow any other node in \( h \), so the probability of an action \( a \in A_h \) being chosen conditional on \( x \in h \) being reached is the same as the probability of \( a \) being chosen conditional on any other \( x' \in h \) being reached.
   d. Formally we define a correspondence \( \kappa_i : \Delta(S_i) \to \Pi_i \) by letting \( \kappa_i(\sigma_i) = \{\pi^i \in \Pi_i | (\ast) \} \) whenever \( h \in H_i \) with \( \sigma_i(\{s_i|s_i \text{ allows } h\}) > 0 \) and \( a \in A(h) \).

**Kuhn’s Theorem:** If \( \sigma_i \in \Delta(S_i) \) and \( \pi^i \in \kappa_i(\sigma_i) \), then \( \sigma_i \) and \( \lambda_i(\pi^i) \) are realization equivalent.

a. The proof is not especially hard, but it is algebra intensive, so it will not be given here. If you feel confident about your ability to handle such calculations you will benefit from trying to construct it.

b. The conclusion of Kuhn’s theorem implies that the probability distributions on $Z$ induced by $\sigma_i$ and $t_{-i}$ is the same as the one induced by $\lambda_i(\pi^i)$ and $t_{-i}$. The probability distribution on $Z$ induced by $\sigma_i$ and $\tau_{-i} \in \Sigma_{-i}$ is the sum over $t_{-i} \in S_{-i}$ of $\prod_{j \neq i} \tau_j(t_j)$ times the probability distribution on $Z$ induced by $\sigma_i$ and $t_{-i}$, and similarly for $\lambda_i(\pi^i)$, so $(\sigma_i, \tau_{-i})$ and $(\lambda_i(\pi^i), \tau_{-i})$ induce the same distribution on $Z$.

c. Kuhn’s theorem has the following immediate corollary: if $\sigma^*$ is a normal form Nash equilibrium, $\pi^* \in \Pi$, and $\pi^*_i \in \kappa_i(\sigma^*_i)$ for all $i$, then $\pi^*$ is a Nash equilibrium of the agent normal form.

i. Specifically, if some information set could improve by deviating, then the corresponding normal form deviation would be an improvement for the agent who chooses at that information set.