Games of Incomplete Information

I. Importance of Information and the General Model

A. Many well known examples illustrate how subtle inferences can occur when agents have different information.

1. There are three white hats and two black hats. There are three agents, each of whom receives a hat from this collection. They are seated in a row.
   a. The first agent says “I can see the hats of the two people in front of me, but not my own hat, and I don’t know what color it is.”
   b. The second agent says “I heard what the first agent said, and can see the hat of the third person in front of me, but I can’t see my own hat, and I don’t know what color it is.”
   c. The third agent says “I can’t see anyone’s hat, but I know what color my hat is.

2. (Aumann) The King of a tropical island announces that there has been some infidelity, and that each day there will be an opportunity for husbands to denounce their unfaithful partners, who will be tossed into the volcano, of course.
   a. When a wife cheats, everyone except her husband quickly learns about it.
b. On the first three days no husbands come forward, but on the fourth day some do. How many and why?

c. What would have happened if the King had not announced (i.e., made it common knowledge) that there was at least one unfaithful wife?

B. The hierarchies of beliefs.

1. Suppose there is a finite set of agents and a set of “physical states of the world,” which may be finite or a compact metric space.

2. Each agent has beliefs concerning the physical states.
   a. Call this the agent’s first order belief.

3. An agent’s second order belief specifies a joint distribution over the cartesian product of the set of physical states and the sets of possible first order beliefs of the other agents.
   a. It is natural to require that the first order belief be the marginal of the second order belief on the space of physical states. Similar consistency conditions can be imposed on the higher order beliefs.

4. An agent’s third order belief specifies a joint distribution over the cartesian product of the set of physical states and the sets of possible second order beliefs of the other agents, and so on.

5. An agent’s type is her infinite sequence of beliefs of all orders.

6. For a long time people thought that this structure was so complicated that it wasn’t even worth writing down, but Mertens and Zamir did write a paper on it, and since then it has become a standard aspect of game theory.
   a. Although infinite dimensional, the space of types is compact because:

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i. The set of probability measures on a compact metric space is a compact metric space.

ii. The cartesian product of any collection of compact metric spaces is compact, if endowed with the product topology.

b. The set of types arising from finite type models of the sort studied below is dense.

c. It can be argued that the topology imposed on the space of types is too coarse for strategic analysis, since certain conditional probabilities are not continuous functions.

II. Harsanyi’s Model of Finite Type Spaces

A. A Bayesian game (in strategic form) is a tuple

\[ G = (I, (S_1, \ldots, S_n), (\Theta_1, \ldots, \Theta_n), (u_1, \ldots, u_n), F) \]

where:

1. \( I = \{1, \ldots, n\} \) is the set of agents.
2. For each \( i \in I \), \( S_i \) is a nonempty finite set of pure strategies. Let \( S = S_1 \times \cdots \times S_n \).
3. For each \( i \in I \), \( \Theta_i \) is a nonempty finite set of types for agent \( i \). Let \( \Theta = \Theta_1 \times \cdots \times \Theta_n \).
4. For each \( i \in I \), \( u_i: S \times \Theta \rightarrow \mathbb{R} \) is a payoff function.
5. \( F \in \Delta(\Theta_1 \times \cdots \times \Theta_n) \) is a Bayesian prior distribution.

B. A (mixed) decision rule for player \( i \) is a function \( \sigma_i: \Theta_i \rightarrow \Delta(S_i) \).

1. The expected utility resulting from a vector \( \sigma = (\sigma_1, \ldots, \sigma_n) \) of decision rules is

\[
\sum_{\theta \in \Theta} \left( \sum_{s \in S} \left( \prod_{i \in I} \sigma_i(s_i | \theta_i) \right) u_i(s, \theta) \right) F(\theta).
\]
2. A Bayesian Nash equilibrium is a vector of decision rules with the property that each agent is maximizing expected utility.

3. Since we may write the expected utility of agent $i$ as

$$\sum_{\theta_i \in \Theta_i} \left( \sum_{\theta_{-i} \in \Theta_{-i}} \left( \sum_{s \in S} \left( \prod_{j \in I} \sigma_j(s_j | \theta_j) \right) u_i(s, \theta) \right) F(\theta_{-i} | \theta_i) \right) F_i(\theta_i)$$

(here $F_i \in \Delta(\Theta_i)$ is the marginal distribution on $\Theta_i$ and $F(\theta_{-i} | \theta_i)$ is the conditional probability of $\theta_{-i}$ given $\theta_i$) we see that maximization for agent $i$ is a matter of maximizing the expected utility of each of her types separately.

4. In this sense, a Bayesian Nash equilibrium is merely a Nash equilibrium of the strategic form game in which each type is regarded as a different agent.

   a. For this reason existence of a Bayesian Nash equilibrium is already guaranteed by the Nash existence theorem.

C. An Example

1. Let $\Theta_1 = \{H, D\}$, let $\Theta_2$ be a singleton, and let $F(H) = .7$ and $F(D) = .3$. Let $S_1 = \{G, B\}$, and $S_2 = \{T, I\}$, and let the payoffs be

$$\begin{pmatrix}
T & I \\
G & ((8, 8), 5) & ((5, 5), 4) \\
B & ((7, 9), 0) & ((0, 2), 3)
\end{pmatrix}$$

where the first pair in each entry gives the payoffs of the types $H$ and $D$ respectively.

2. The interpretation is that agent 1 is a supplier who can choose to deliver either good or bad products to agent 2, the customer. Agent 1 is either honest (type $H$) in which case it is a dominant strategy to supply good products, or dishonest (type $D$) in which case the
gain resulting from getting away with supplying bad goods outweighs whatever feelings of guilt this entails. Agent 2 can either trust agent 1 or inspect the goods before accepting them, which is costly for all concerned.

3. In any Bayesian Nash equilibrium the honest type will supply good products. If the dishonest type also supplies good products then the best response for agent 2 is to never inspect, in which case the dishonest type does better to supply bad products. On the other hand if the dishonest type always supplies bad products, then the best response of agent 2 is to always inspect, since the gain of 3 with probability .3 outweighs the loss of 1 with probability .7, and if inspection is expected the dishonest type prefers to deliver good products. The only Bayesian Nash equilibrium has agent 2 mixing between trust and inspection in a way that makes the dishonest type indifferent, which means that inspection will occur with probability .25. The dishonest type must mix in a way that leaves agent 2 indifferent, which means that the posterior probability of good products must be 3/4.

III. Harsanyi Purification

A. A *purification theorem* is a result asserting that (possibly after some modification of the model) there is a Nash equilibrium in pure strategies.

1. People rarely randomize explicitly.

2. When there are many agents whose interactions are impersonal (I care about what I do and the distribution of other agents’ actions, but not what any other individual does) it seems that agents should have little reason to randomize.

B. The *purification theorem* of Harsanyi explains how mixed strategy equilibria (as perceived by “the social scientist”) might occur with only a minute
amount of randomization. If actual payoffs differ from the payoffs as described by the model according to small and independently distributed perturbations, with each agent’s perturbation known by that agent, but not by others, then a pure equilibrium of the underlying Bayesian game will resemble a mixed equilibrium of the model.

1. Starting with a normal form game $G = (I, S_1, \ldots, S_n, u_1, \ldots, u_n)$ we form a Bayesian game

$$\tilde{G} = (I, (S_1, \ldots, S_n), (\Theta_1, \ldots, \Theta_n), (\tilde{u}_1, \ldots, \tilde{u}_n), F)$$

with the following properties.

a. $\Theta_1, \ldots, \Theta_n$ each have many elements.

b. $F = F_1 \otimes \cdots \otimes F_n$ is a product of distributions on the $\Theta_i$, which is to say that $\theta_1, \ldots, \theta_n$ are independently distributed.

c. We require that each number $\tilde{u}_i(s, \theta)$ is only slightly different from $u_i(s)$.

i. With these assumptions, each equilibrium of $\tilde{G}$ will be an approximate equilibrium of $G$.

ii. If we also require that the expectations

$$\hat{u}_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \tilde{u}_i(s, (\theta_i, \theta_{-i}))(\prod_{j \neq i} F_j(\theta_j))$$

are slightly different, then all equilibria of $\tilde{G}$ have very little mixing.

IV. Correlated Equilibrium

A. This framework allows a simple explanation of the notion of correlated equilibrium, along the lines of Aumann’s “Correlated Equilibrium as an Expression of Bayesian Rationality” (Econometrica 1987).
B. Starting with a normal form game $G = (I, S_1, \ldots, S_n, u_1, \ldots, u_n)$ we form a Bayesian game

$$\tilde{G} = (I, (S_1, \ldots, S_n), (\Theta_1, \ldots, \Theta_n), (\tilde{u}_1, \ldots, \tilde{u}_n), F)$$

such that $\tilde{u}_i(s, \theta) = u(s)$ for all $i, s \in S$, and $\theta \in \Theta$.

C. A correlated equilibrium of $G$ is a $\tau \in \Delta(S)$ such that for some such $\tilde{G}$ there is a Bayesian Nash equilibrium $\sigma$ such that

$$\tau(s) = \sum_{\theta \in \Theta} F(\theta) \prod_{i} \sigma_i(s_i|\theta_i)$$

for all $s \in S$.

D. Correlated equilibrium may be thought of as describing the set of credible agreements the agents could make about how to play the game if they have a trusted mediator. The idea is that the mediator randomizes over $S$ according to $\tau$. When $s$ is drawn, the mediator asks each $i$ to play $s_i$, and it is rational to comply because $s_i$ is a best response to the beliefs induced by being asked to play $i$.

E. Consider the following example due to Aumann:

$$\begin{pmatrix}
1 & 2 \\
L & R \\
U & (6, 6) \\
D & (7, 1)
\end{pmatrix}$$

1. In the unique Nash equilibrium both agents assign equal probability to the two pure strategies, receiving an expected payoff of $3\frac{1}{2}$.

2. There is the correlated equilibrium shown below which gives each agent a payoff of $4\frac{2}{3}$:

$$\begin{pmatrix}
1 & 2 \\
L & M \\
U & \left(\frac{1}{3}, \frac{1}{3}\right) \\
D & \left(\frac{1}{3}, 0\right)
\end{pmatrix}$$
V. Global Games

A. Many aggregate social phenomena involve situations in which multiple equilibria exist.
   1. If there is going to be a bank run, everyone’s optimal action is to go to the bank and withdraw their money.
   2. In a speculative currency attack, speculators short the currency, hoping to force the government to devalue, after which they can cover their positions inexpensively.
   3. If everyone joins in a revolution against a tyrannical regime, it will succeed with high probability, but those who participate will be punished if the revolution fails.

B. The theory of global games looks at these situations from a different point of view, in which the agents observe signals that are correlated with an underlying state that affect the payoffs.

C. (Carlsson and van Damme) Two players each have to decide whether to invest $I$ or not $N$ in a risky project.
   1. In the basic model there is a state of the world $\omega \in \mathbb{R}$. Conditional on this the payoffs are as follows:

      \[
      \begin{pmatrix}
      1 \backslash 2 & I & N \\
      I & (\omega, \omega) & (\omega - 1, 0) \\
      N & (0, \omega - 1) & (0, 0)
      \end{pmatrix}
      \]

      a. If $\omega \geq 1$ it is a dominant strategy to invest.
      b. If $0 < \omega < 1$ there are two pure equilibria and a mixed equilibrium.
      c. If $\omega \leq 0$ it is a dominant strategy to not invest.
   2. Now suppose that instead of observing $\omega$, each agent $i$ observes $x_i = \omega + \varepsilon_i$ where $\varepsilon_1, \varepsilon_2$ are i.i.d. normally distributed random variables
with mean 0 and variance $\sigma^2$.

a. Now it is an equilibrium for each agent $i$ to invest if $x_i > \frac{1}{2}$ and to not invest if $x_i < \frac{1}{2}$.

b. Rather complex arguments show that this is the unique equilibrium.

D. The point is that in this model there is also a bank run (say) sometimes and not others, it is more tractable in the sense that there is a unique equilibrium, and in addition it makes the prediction that the probability of a bank run is positively correlated with the stress on the bank.

1. Markus Brueckner has a paper in *Econometrica* showing that during a wave of democratizing revolutions in Africa between 1990 and 1995, the revolution in each country was often during a year of low rainfall and consequent poor harvest.

**VI. Introduction to Auctions**

A. Auctions are an ancient institution.

1. One would like to explain this stability as a consequence of optimality.

B. Many markets utilize auctions.

1. Mineral rights.
2. Slaves.
3. Art.
4. Flowers.
5. Wine.
7. State owned businesses.

C. Questions.

1. Equilibrium strategies and outcomes.
2. Optimality.
a. Pareto.
b. For the seller.

3. Do auction outcomes resemble market outcomes?
4. Aggregation of information.

VII. Types of Auction.
A. Dutch – the price descends continuously until a bidder yells “stop”.
B. First price sealed bid – the highest bidder pays his or her bid.
C. Second price sealed bid – the highest bidder pays the second highest bid.
D. English – bids are announced until there is a bid that no one wants to top.

VIII. Models of Valuation
A. Each agent observes a signal $\theta_i$, which we now assume to be a real number.
B. Each player $i = 1, \ldots, I$ has a value $u_i(\theta_1, \ldots, \theta_I)$ for the object and is risk neutral with respect to money, so that the payoff is either $u_i(\theta_1, \ldots, \theta_I) - p_i$ if agent $i$ wins or $-p_i$ if the agent does not, where $p_i$ is the payment from the agent to the auctioneer.

1. In most auctions $p_i = 0$ when agent $i$ does not win, but some attention has been given to “all pay auctions” in which each agent pays her bid regardless of whether she wins.
C. The distribution of signal vectors $(\theta_1, \ldots, \theta_I)$ is assumed to be symmetric.
D. There is $u : \mathbb{R}^I \to \mathbb{R}$ such that for all $i$ and $\theta$,

$$u_i(\theta) = u(\theta_i, \{\theta_j\}_{j \neq i})$$

1. Assumption: $u$ is nonnegative, continuous, and nondecreasing.
2. Assumption: $\mathbb{E}(u_i) < \infty$.

E Example: The Independent Private Values Model.
1. $u_i(\theta_1, \ldots, \theta_I) = \theta_i$. 

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2. \( \theta_1, \ldots, \theta_I \) are independent and identically distributed.

F. Example: The Mineral Rights Model.

1. The unknown “true value” \( v \) is the same for everyone.
2. Conditional on \( v \), \( \theta_i \) and \( \theta_j \) are statistically independent. (For example it might be the case that \( \theta_i = v + \epsilon_i \) where \( \epsilon_1, \ldots, \epsilon_I \) are i.i.d. normal random variables.)
3. In this setting one has the so-called “winner’s curse” – the expected value of the object conditional on \( \theta_i \) may be much higher than the expected value conditional on \( \theta_i \) and \( \theta_i = \max\{\theta_j\} \).

IX. Symmetric Equilibrium in Second Price Auctions.

A. We assume that all valuations are monetary and that all agents are risk neutral.

B. We assume that agents \( j \neq i \) are following the same bidding strategy that is a monotonic increasing function \( b : \theta_j \mapsto b(\theta_j) \), and we analyze the behavior of agent \( i \).

1. Let \( Y_1 = \max_{j \neq i} \theta_j \) be the top order statistic of the other agents’ signals.
2. Agent i’s problem is \( \max \mathbb{E}[(u_i - b(Y_1)) \cdot 1_{b(Y_1) < b}(\theta_i)] \).

C. Set \( v(\theta, y) = \mathbb{E}[u_i|\theta_i = \theta, Y_1 = y] \). Let \( b^*(\theta) = v(\theta, \theta) \).

Theorem: If the model of valuation is such that \( v \) is nondecreasing in both variables, then \( (b^*, \ldots, b^*) \) is an equilibrium.

Proof: The consequences of, say, increasing the bid over \( b^*(\theta_i) \) to \( b \) is that one wins in situations where one would have lost before, namely when \( Y_1 \) is between \( \theta_i \) and \( b^{*-1}(b) \).

Since, for such \( Y_1 \), \( v(\theta_i, Y_i) \leq v(b^{*-1}(b), b^{*-1}(b)) = b \), this is not good. A similar argument shows that one does not do better by bidding less than \( b^*(\theta_i) \).
X. Symmetric Equilibrium in First Price Auctions.

A. Again we wish to determine when \((b^*, \ldots, b^*)\) is an equilibrium. Assume \(b^*\) is increasing and differentiable.

B. Assume all \(j \neq i\) play \(b^*\) while bidder \(i\) observes \(\theta_i\) and bids \(b\). Her expected payoff is

\[
\Pi_i(b; \theta_i) = E[(u_i - b) \cdot 1_{\{b^*(Y_i) < b\}} | \theta_i]
\]

\[
= E[E[(u_i - b) \cdot 1_{\{b^*(Y_i) < b\}} | \theta_i, Y_i] | \theta_i]
\]

\[
= E[(v(\theta_i, Y_i) - b) \cdot 1_{\{b^*(Y_i) < b\}} | \theta_i]
\]

\[
= \int_0^{b^*-1(b)} (v(\theta_i, \alpha) - b) f_{Y_i}(\alpha | \theta_i) d\alpha
\]

Here \(f_{Y_i}(\cdot | \theta_i)\) is the density of \(Y_i\) given \(\theta_i\). Let \(F_{Y_i}(\cdot | \theta_i)\) be the cumulative distribution function.

C. If \(b = b^*(\theta)\) is optimal then (after some work) the first order condition yields the differential equation

\[
b^{*\prime}(\theta) = (v(\theta, \theta) - b^*(\theta)) \left[ \frac{f_{Y_i}(\theta | \theta)}{F_{Y_i}(\theta | \theta)} \right]
\]

1. Necessarily \(v(\theta, \theta) - b^*(\theta)\) is nonnegative for all \(\theta\), since otherwise one is bidding when one prefers to not win.

2. Let \(\underline{\theta}\) be the lowest possible signal. If \(v(\theta, \underline{\theta}) - b^*(\underline{\theta}) > 0\) then one does better by replacing \(b^*(\underline{\theta})\) with \(b^*(\underline{\theta}) + \varepsilon\), so

\[
b^*(\underline{\theta}) = v(\theta, \underline{\theta}).
\]

XI. A More General Valuation Model

A. For some markets neither the independent private values model nor the mineral rights model seems appropriate. For instance consider art.

1. The IPV model fails to take into account that the investment value of a painting will be positively correlated with others’ valuations.
2. The MR model fails to consider individual tastes.
3. Milgrom and Weber (1982) construct a model that has both IPV and MR as special cases.

B. Variables.
1. \( X = (X_1, \ldots, X_n) \) – real valued informational variables, one for each bidder.
2. \( S \) – another random variable, possibly observed by the seller.
3. \( V_i = u_i(S, X) \) – the full information value to bidder \( i \).

C. Assumptions
1. **Assumption 1:** There is \( u : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) such that

   \[
u_i(S, X) = u(S, X_i, \{X_j\}_{j \neq i})\]

2. **Assumption 2:** \( u \) is nonnegative, continuous, and nondecreasing.
3. **Assumption 3:** \( E(V_i) < \infty \).
4. **Assumption 4:** \((S, X_1, \ldots, X_n)\) are jointly distributed with density \( f \), where \( f \) is symmetric in its last \( n \) arguments.
5. **Assumption 5:** \( S \) and \( X_1, \ldots, X_n \) are affiliated.

**Definition:** For \( z, z' \in \mathbb{R}^{n+1} \), let

\[
z \vee z' = (\max\{z_0, z'_0\}, \ldots, \max\{z_n, z'_n\}) \quad \text{and let} \quad z \land z' = (\min\{z_0, z'_0\}, \ldots, \min\{z_n, z'_n\}).
\]

The density \( f \) is **affiliated** if, for almost all \( z, z' \),

\[
f(z \land z')f(z \vee z') \geq f(z)f(z').
\]

D. Connections with other models.
1. In the IPV model we have \( f(z \land z')f(z \vee z') = f(z)f(z') \).
2. In the MR model we have \( f(s, x) = h(s)g(x_1|s) \ldots, g(x_n|s) \) where \( g \) satisfies the monotone likelihood ratio property:

\[
g(x|s)g(x'|s') \geq g(x|s')g(x'|s), \quad \text{or} \quad \frac{g(x|s)}{g(x'|s)} \geq \frac{g(x|s')}{g(x'|s')},
\]

when \( s' > s \) and \( x' > x \).

XII. Equilibrium in Second Price Auctions.

A. We assume that all valuations are monetary and that all agents are risk neutral.

B. We assume that agents \( j \neq 1 \) are following bidding strategies \( b_j : X_j \mapsto b_j \), and we analyze the behavior of agent 1.

1. Set \( W = \max_{j \neq i} b_j(X_j) \).

2. Agent 1’s problem is \( \max E[(V_1 - W) \cdot 1\{W < b\}]|X_1] \).

C. Set \( v(x, y) = E[V_1|X_1 = x, Y_1 = y] \). By the Theorems above, \( v \) is nondecreasing in both variables. Let \( b^*(x) = v(x, x) \).

Theorem 6: \((b^*, \ldots, b^*)\) is an equilibrium.

Proof. The consequences of, say, increasing the bid over \( b^* \) is that one wins in situations where one would have lost before. If \( Y_1 = X \) and everybody else is following \( b^* \), one can only do worse by departing from \( b^* \).

XIII. Equilibrium in English Auctions.

A. There are many variants of the English auction. We consider one in which the information people have is easily specified. Each bidder has a button, and at the beginning all bidders are depressing their buttons. As time passes the posted price rises continuously, and a bidder drops out by releasing the button. The remaining bidders know how many bidders have dropped out and the prices at which they departed. The winner is the last bidder, and he or she pays the price at which the last bidder departed.
B. A strategy is described by functions $b_{ik}(x_i|p_1, \ldots, p_k)$, $p_1 \leq \ldots \leq p_k$ describing the price at which agent $i$ drops out when his signal is $x_i$ and $k$ other bidders drop out at prices $p_1, \ldots, p_k$.

C. Define the functions $b^*_k$ inductively by

$$b^*_0(x) = E[V_1|X_1 = x, Y_1 = x, \ldots, Y_{n-1} = x]$$

and

$$b^*_k(x|p_1, \ldots, p_k) = E[V_1|X_1 = x, Y_1 = x, \ldots, Y_{n-k-1} = x],$$

$$b^*_k(Y_{n-k}|p_1, \ldots, p_{k-1}) = p_k, \ldots, b^*_0(Y_{n-1}) = p_1].$$

1. Let $b^* = (b^*_0, \ldots, b^*_{n-2})$.

**Theorem 10:** $(b^*, \ldots, b^*)$ is an equilibrium of the button auction.

*Proof.* If all other agents follow $b^*$, it is again the case that by following $b^*$ agent 1 wins if and only if that is what he wants to do conditional on others’ information. ■