I. Significance of Game Theory

A. Limitations of traditional economic theory.
   1. In perfect competition and monopoly, all agents (except perhaps one) face a decision problem characterized by macro variables.
   2. During the last 35 years game theory has become a methodology that is at least as important as price theory in contemporary research. There are too many examples to list.

B. How does game theory proceed, and how should we understand the relationships between the concepts we will study and phenomena in the world? Here is a simple scheme:
   1. Pose a mathematically exact description of the physical environment and the agents’ preferences.
   2. Specify a domain of applications.
   3. Based on the character of the applications, argue for a solution concept, meaning a set of predicted restrictions on behavior.
   4. Test conceptually.
      a. Does the solution concept give “good” answers in examples?
      b. Does it clash with, or complement, other solution concepts?
      c. Can it be given a compelling axiomatic foundation?
5. Test empirically.
   a. Is the model predictively accurate?
   b. Does the model at least fit some applications, even if there are others where its predictions fail?
   c. Does the model “convey understanding” in perhaps an even looser sense?

C. Criticisms.
1. The model is always much simpler than the reality.
2. Economists tend to downgrade direct “observation” of preferences, say by introspection. If preferences cannot be observed, but only revealed, the theory lacks content.
3. We cling to “coherent” theories in spite of empirical failures.

II. Normal Form Games

A. Formalism of finite normal (strategic) form games.
1. The set of players \( \{1, \ldots, I\} \).
2. For each \( i = 1, \ldots, I \), a finite nonempty set \( S_i \) of pure strategies. The set of pure strategy vectors is \( S := S_1 \times \ldots \times S_I \).
3. For each \( i = 1, \ldots, I \), a utility or payoff function \( u_i : S \to \mathbb{R} \), which is interpreted as a von Neumann-Morgenstern utility.
4. Usually the normal form is thought of as being derived from an extensive form game.

B. The One-Shot Interpretation.
1. The game is conceived as the entire world.
2. This avoids psychologically natural confusion between the given game and the game obtained by playing the given game repeatedly.
   a. Consider the two-fold repetition of

\[
\begin{array}{cccc}
L \quad & R \\
U & \begin{pmatrix} \langle 6, 6 \rangle & \langle 1, 7 \rangle \end{pmatrix} \\
D & \begin{pmatrix} \langle 7, 1 \rangle & \langle 0, 0 \rangle \end{pmatrix}
\end{array}
\]
3. This interpretation also avoids reputation effects having to do with the consequences of others observing behavior in the game.

   a. In principle, one cannot do science on a single data point.
   b. The agents have no experiential basis for forming beliefs about each other.

C. The Repeated-Anonymous Interpretation.
   1. The game happens repeatedly, and everyone observes how the game is played on average, but no individual’s behavior in any play is remembered.
   2. This gives a strategic interpretation of cultural differences. “When in Rome do as the Romans do.”
   3. Critique.
      a. This interpretation depends on some notion of “similarity” between games that is to some extent arbitrary.
      b. Kreps’ *Game Theory in Economic Modelling* has an extensive discussion of factors that might give rise to an “obvious” or “focal” way of playing a game.

D. The Evolutionary Interpretation.
   1. Strategies are viewed as genetically determined, rather than rationally calculated.
   2. Unquestionably valid in biology, subject to data limitations, but specifies a version of rationality that is very bounded.

III. Introduction to Expected Utility

A. In the remainder of this lecture we study the von Neumann-Morgenstern expected utility theory of choice in the face of uncertainty.
   1. Although this is controversial, after this lecture we will follow the usual practice of microeconomists in using this as our foundational theory.
   2. In addition to its importance in game theory, it is also the standard
foundation for information economics and financial economics.

B. Modelling uncertainty.

1. A standard point of view in statistics is that there is an underlying set $\Omega$ of states of the world which has a probability measure $P$.
   a. In principle this should be a general measure space.
   b. Although we will eventually study some random variables that are continuously distributed, mostly it will be enough to treat $\Omega$ as a finite set. In general, for a finite set $X$ the set of probability measures on $X$ is
   \[
   \Delta(X) = \{ \sigma : X \rightarrow [0, 1] : \sum_{x \in X} \sigma(x) = 1 \}.
   \]
   a. Let $\Xi$ be the set of nonempty subsets of $\Omega$. A conditional probability function is a function $p : \Xi \rightarrow \Delta(\Omega)$ such that
   \[
   \sum_{S \in \Xi} p(S \mid S) = 1
   \]
   for all $S \in \Xi$. We say that $p$ is Bayesian if
   \[
   p(R \mid T) = p(R \mid S)p(S \mid T)
   \]
   for all $R \subset S \subset T \subset \Omega$ such that $S \neq \emptyset$.

2. A variable that is uncertain is modelled as a function from $\Omega$ to the set in which the variable takes values.

3. Uncertainty is a property of the knowledge of some individual. In game theory there are multiple individuals, so there is already a potential issue here.

4. The standard point of view, which we adopt unless there is an explicit statement to the contrary, is that $(\Omega, P)$ expresses the uncertainty of the “social scientist.” The uncertainty of each agent in the model is derived from this by combining the social scientist’s initial perspective with additional information.
   a. Already this perspective imposes what is known as the common prior assumption, which is a genuine restriction that is somewhat controversial.
C. Expected Utility

1. Let $A$ be a finite set of outcomes.

2. A simple lottery is a probability measure on $A$.
   a. Thus we might write $\frac{1}{3}a + \frac{1}{2}b + \frac{1}{6}c$.
   b. We will also sometimes consider compound lotteries which assign a total of one unit of probability to outcomes and other lotteries, which may be either simple or compound, e.g., $\frac{1}{3}(\frac{2}{3}a + \frac{2}{3}b) + \frac{1}{2}b + \frac{1}{6}c$.

3. A von Neumann-Morgenstern utility function is a function $u : A \rightarrow \mathbb{R}$.

4. The von Neumann-Morgenstern theory of choice under uncertainty amounts to the following hypotheses:
   a. The agent is indifferent between any compound lottery and the equivalent simple lottery.
   b. The agent weakly prefers simple lottery $f$ to simple lottery $g$ if and only if
   \[
   \sum_{a \in A} f(a)u(a) \geq \sum_{a \in A} g(a)u(a).
   \]
   The quantity $\sum_{a \in A} f(a)u(a)$ is the expected utility resulting from lottery $f$.
   c. Remark: In some sense Daniel Bernoulli had already invented the notion of expected utility, in response to the St. Petersburg paradox. (“What should you be willing to pay for the lottery $\frac{1}{2}$\$2 + \frac{1}{4}$\$4 + \frac{1}{8}$\$8 + \cdots$?”) Certainly von Neumann and Morgenstern gave an important formal treatment of the concept, but the extent of conceptual novelty has never been clear to me.

D. The vNM expected utility theory of choice under uncertainty makes strong predictions. There are a number of circumstances where these predictions are systematically violated.

1. The Allais paradox:
   a. Choose between the lotteries $0.11 \times \$5,000,000 + 0.89 \times \$0$ and
.10 × $20,000,000 + .90 × $0.

b. Choose between $5,000,000 for sure and the lottery and .89 × $5,000,000 + .10 × $20,000,000 + .01 × $0.

2. The Ellsberg paradox: an urn has 90 balls, 30 of which are red, and the rest of which are yellow or black. One ball will be drawn.
   a. Choose between receiving $100 if a red ball is drawn or receiving $100 is a black ball is drawn.
   b. Choose between receiving $100 if a red or yellow ball is drawn or receiving $100 is a black or yellow ball is drawn.

3. The Rabin paradox: people are unreasonably risk averse with respect to small gambles, for instance declining a gamble that loses $100 half the time and wins $101 half the time.
   a. To make this example convincing it is important to avoid issues of convenience such as going to the ATM sooner than one would otherwise, and also to avoid settings (e.g., graduate school) in which the decision maker might be credit constrained.

4. Framing effects: in an emergency or medical situation you must choose between A and B, where:
   a. A saves 200 lives for sure, and B saves 600 lives with probability \( \frac{1}{3} \) and saves no one with probability \( \frac{2}{3} \).
   b. A results in 400 deaths for sure, while B result in a \( \frac{1}{3} \) chance that no one will die and a \( \frac{2}{3} \) chance of 600 deaths.
   c. Framing effects affect many types of decision making, and are thus a more general challenge to rationality.

5. With respect to all of these concerns, one may argue that for any scientific theory, being “true” in some literal minded sense is not the same thing as being useful, in the sense of leading to illuminating insights or serving as a foundation for larger theories.

E. Normative interpretation.

1. The vNM theory may be viewed as prescriptive rather than descrip-
tive, a model of how people would like to behave, or how they think they or others should behave.

2. With respect to the paradoxes above, one may inquire whether the effect persists once the decision maker understands it. If the decision maker changes her choices once she understands the nature of the paradox, then they seem akin to optical illusions rather than genuine preferences.

IV. Axiomatic Characterization

A. We now present an axiomatic characterization of expected utility. This has some mathematical uses, but more importantly it illuminates what is involved in accepting or rejecting expected utility.

B. We follow Myerson, since his framework is more general and entails a derivation of subjective probabilities.

1. $A$ is a finite set of outcomes.
2. $\Omega$ is a finite set of states of the world.
3. A lottery is a function $f : \Omega \rightarrow \Delta(X)$.

   a. Thus we are studying the decision maker's subjective beliefs concerning the probabilities of various states of the world by looking at his preferences over lotteries that assign objective probabilities to each outcome in each state of the world.

4. We write $f \succeq_S g$ to indicate that the decision maker prefers lottery $f$ to lottery $g$ when she has the additional information that the state of the world is an element of $S \in \mathcal{S}$. Thus our given data is a system of binary relations on the set of lotteries, one for each nonempty set of states of the world. As usual we write $f \succ_S g$ to indicate $f \succeq_S g$ and not $g \succeq_S f$, and we write $f \sim_S g$ to indicate that $f \succeq_S g$ and $g \succeq_S f$.

C. The axioms:

1. (Completeness) For all lotteries $f$ and $g$ and all $S \in \mathcal{S}$, either $f \succeq_S g$ or $g \succeq_S f$.
2. (Transitivity) If $f \succeq_S g$ and $g \succeq_S h$, then $f \succeq_S h$. 


3. (Relevance) If \( f(\cdot|s) = g(\cdot|s) \) for all \( s \in S \), then \( f \sim_S g \).

4. (Monotonicity) If \( f \succ_S g \) and \( 0 \leq \alpha < \beta < 1 \), then \( \beta f + (1 - \beta)g \succ_S \alpha f + (1 - \alpha)g \).

5. (Continuity) If \( f \succeq_S g \) and \( g \succeq_S h \), then there is a number \( \alpha \in [0,1] \) such that \( g \sim_S \alpha f + (1 - \alpha)h \).

6. (Objective Substitution) If \( e \succeq_S f \), \( g \succeq_S h \), and \( 0 \leq \alpha \leq 1 \), then \( \alpha e + (1 - \alpha)g \succeq_S \alpha f + (1 - \alpha)h \).

7. (Strict Objective Substitution) If \( e \succ_S f \), \( g \succeq_S h \), and \( 0 < \alpha \leq 1 \), then \( \alpha e + (1 - \alpha)g \succ_S \alpha f + (1 - \alpha)h \).

8. (Subjective Substitution) If \( f \succeq_S g \), \( f \succeq_T g \), and \( S \cap T = \emptyset \), then \( f \succeq_{S\cup T} g \).

9. (Strict Subjective Substitution) If \( f \succ_S g \), \( f \succ_T g \), and \( S \cap T = \emptyset \), then \( f \succ_{S\cup T} g \).

10. (Revealed Improbability) If \( e \succ_R f \), \( e|_S = f|_S \), \( R \cap S = \emptyset \), \( e \sim_{R\cup S} f \), and \( S \subseteq T \subseteq \Omega \), then \( g \succeq_T h \) if and only if \( g \succeq_{T\cup R} h \) for all lotteries \( g \) and \( h \).

11. (State Independence) For any \( s, t \in \Omega \), if \( f(\cdot|s) = f(\cdot|t), g(\cdot|s) = g(\cdot|t) \), and \( f \succeq_{\{s\}} g \), then \( f \succeq_{\{t\}} g \).

**Remark:** We have left out Myerson’s axiom “Interest” in order to get a more “principled” result. Without this axiom, we need to add Revealed Improbability in order for the result below to hold, as we show now by example. We will say that a state \( s \in \Omega \) is inert if \( a \sim_{\{s\}} b \) for all \( a, b \in A \). We say that \( s \) is active if it is not inert. Consider states \( s \) and \( t \) with \( s \) active. We say that \( s \) is infinitely less likely than \( t \) if \( f \sim_{\{s,t\}} g \) for all lotteries \( f \) and \( g \) such that \( f(t) = g(t) \).

**Example:** Suppose \( A = \{a, b\} \) and \( \Omega = \{r, s, t\} \). Suppose that \( t \) is inert, while \( a \) is strictly preferred to \( b \) at states \( r \) and \( s \). Suppose that \( t \) is infinitely more likely than \( s \) and \( s \) is infinitely more probable than \( r \). Suppose that \( f \succeq_{\{r,t\}} g \) if and only if \( f(r) \) assigns at least as much probability to \( a \) as \( g(r) \). Finally, suppose that \( f \sim_{\Omega} g \) for all \( f \) and \( g \). There is no expected utility representation of these preferences, but they satisfy axioms
1-7 because these axioms refer to a single $S \subset \Omega$. They also satisfy axioms 8 and 9, as you can easily verify by considering all relevant cases. (Note that axiom 8 can only be violated if $f \succeq_{S \cup T} g$ is possible and axiom 9 can only be violated if both $f \succ_S g$ and $g \succ_T g$ are possible.)

**Theorem:** If the first ten axioms above hold, then there is a utility function $u : X \times \Omega \to \mathbb{R}$ and a Bayesian conditional probability function $p : \Xi \to \Delta(\Omega)$ such that for all lotteries $f$ and $g$ and all $S \in \Xi$,

$$f \succeq_S g \quad \text{if and only if} \quad \sum_{s \in S} p(s|S) \sum_{a \in A} f(a|s)u(a, s) \geq \sum_{s \in S} p(s|S) \sum_{a \in A} g(a|s)u(a, s).$$

If, in addition, State Independence holds, we may insist that $u$ is a function only of the outcome.

**Proof.** We will first find a utility function $u(\cdot, s)$ for a given state $s$. In view of Completeness, Transitivity, and Relevance, $\succeq_{\{s\}}$ may be regarded as a preference relation on $\Delta(A)$.

If $s$ is inert, repeated applications of Objective Substitution shows that $\succeq_{\{s\}}$ is indifferent between all elements of $\Delta(A)$, and we let $u(\cdot, s) : A \to \mathbb{R}$ be identically zero.

Suppose that $s$ is active. Then there is an $s$-best element of $A$, say $b_s$, and an $s$-worst element of $A$, say $w_s$. Let $u(b_s, s) = 1$ and $u(w_s, s) = 0$. For each $a \in A$, Continuity gives a number $\alpha_a \in [0, 1]$ such that $a \sim_{\{s\}} \alpha_a b_s + (1 - \alpha_a) w_s$. Strict Objective Substitution implies that $\alpha_a b_s + (1 - \alpha_a) w_s \succ_{\{s\}} \alpha_a b_s + (1 - \alpha_a) w_s$ if $1 \geq \alpha > \alpha_a$ and $\alpha_a b_s + (1 - \alpha_a) w_s \succ_{\{s\}} \alpha_a b_s + (1 - \alpha_a) w_s$ if $\alpha_a > \alpha \geq 0$. Thus there is a unique such $\alpha_a$, and we set $u(a, s) = \alpha_a$. If State Independence holds, then either all states are inert or all states are active, and in either case $u$ is a function of $a$.

Let $M$ be a lottery such that for each active $s$, $M(s)$ assigns all probability to $b_s$, and let $m$ be a lottery such that for each active $s$, $m(s)$ assigns all probability to $w_s$. Repeated applications of Strict Subjective Substitution show that $M \succ_S m$ if $S \in \Xi$ contains only active states. Repeated applications of Subjective Substitution show that $M \succeq_S f \succeq_S m$ for all $S \in \Xi$ and all lotteries $f$. For each $S \in \Xi$ let $b_S$ be a lottery such that $b_S(s)$ assigns all probability to $b_s$ if $s \in S$ is active, and $b_S(s)$ assigns all probability to $w_s$ if $s \in \Omega \setminus S$ is active.
For $s \in \Omega$ and $\sigma \in \Delta(A)$ let
\[
E_\sigma(u|\{s\}) = \sum_{a \in A} \sigma(a)u(a, s).
\]
Repeated applications of Objective Substitution imply that $f \sim_{\{s\}} g$ when $E_{f(s)}(u|\{s\}) = E_{g(s)}(u|\{s\})$. If $s$ is active, then $M \succ_{\{s\}} m$, and Continuity implies that $f \succ_{\{s\}} \alpha M + (1 - \alpha)m$ if $\alpha < E_p(u(f)|\{s\})$ and $\alpha M + (1 - \alpha)m \succ_{\{s\}} f$ if $\alpha > E_p(u(f)|\{s\})$. That is, the expected utility hypothesis holds on each state.

For any $T \in \Xi$ whose elements are all active and any $S \subset T$ there is (by Continuity) some $p(S|T) \in [0, 1]$ such that
\[
b_S \sim_T p(S|T)M + (1 - p(S|T))m,
\]
and Monotonicity implies that there is a unique such number. Suppose that $R \subset S \subset T$, and all elements of $T$ are active. Since $M$ and $b_S$ agree on $S$, Relevance gives
\[
b_R \sim_S p(R|S)b_S + (1 - p(R|S))m
\]
Since $b_R$, $b_S$, and $m$ all agree outside of $S$ we have
\[
b_R \sim_{T \setminus S} p(R|S)b_S + (1 - p(R|S))m.
\]
Using Subjective Substitution,
\[
b_R \sim_T p(R|S)b_S + (1 - p(R|S))m.
\]
Using Objective Substitution, then algebra, gives
\[
b_R \sim_T p(R|S)(p(S|T)M + (1 - p(S|T))m) + (1 - p(R|S))m
\]
\[= p(R|S)p(S|T)M + (1 - p(R|S)p(S|T))m.
\]
Therefore $p(R|T) = p(R|S)p(S|T)$ when all states in $T$ are active.

The next portion of the proof studies the “is infinitely less likely than” relation. First, we claim that if $s$ and $t$ are active, then $s$ is infinitely less likely than $t$ if and only
if \( p(s|\{s, t\}) = 0. \) If \( p(s|\{s, t\}) > 0, \) then from the definition of \( p \) there are lotteries \( f \) and \( g \) with \( f(t) = g(t) \) and \( f \succ_{\{s, t\}} g. \) If \( p(s|\{s, t\}) = 0, \) then Subjective Substitution implies, first, that \( f \sim_{\{s, t\}} b_{\{s\}} \) for all lotteries \( f \) that agree with \( m \) on \( t, \) and then that \( f \sim_{\{s, t\}} g \) whenever \( f(t) = g(t). \)

We now suppose that \( s \) is an active state that is not infinitely less likely than \( t. \) We claim that if \( f \) and \( g \) are lotteries with \( f(t) = g(t), \) then \( f \succ_{\{s\}} g \) if and only if \( f \succ_{\{s, t\}} g. \) Relevance implies that \( f \sim_{\{t\}} g, \) so if \( f \succ_{\{s\}} g, \) then Subjective Substitution implies that \( f \succ_{\{s, t\}} g. \) To prove that \( f \succ_{\{s\}} g \) implies \( f \succ_{\{s, t\}} g \) we develop a special case of what will be the final step in the argument. Let \( k_f \) be a lottery such that \( k_f(s) \) assigns probability \( E_f(s)(u|\{s\}) \) to \( b_s \) and probability \( 1 - E_f(s)(u|\{s\}) \) to \( w_s, \) and \( k_f(t) \) assigns probability \( E_f(t)(u|\{t\}) \) to \( b_t \) and probability \( 1 - E_f(t)(u|\{t\}) \) to \( w_t. \) In view of the definition of \( u, \) repeated applications of Objective Substitution implies that \( f \sim_{\{t\}} k_f \) for all \( t, \) after which Subjective Substitution implies that \( f \sim_{\{s, t\}} k_f. \) Let

\[
E_p(u(f)|\{s, t\}) = p(s|\{s, t\})E_f(s)(u|\{s\}) + p(t|\{s, t\})E_f(t)(u|\{t\}).
\]

Let \( \ell_f \) be \( E_p(u(f)|\{s, t\}) \) times \( M \) plus \( 1 - E_p(u(f)|\{s, t\}) \) times \( m. \) In view of the definition of \( p, \) repeated applications of Objective Substitution imply that \( k_f \sim_{\{s, t\}} \ell_f, \) so \( f \sim_{\{s, t\}} \ell_f. \)

We have \( M \succ_{\{s, t\}} m, \) and Continuity implies that \( f \succ_{\{s, t\}} \alpha M + (1 - \alpha)m \) if \( \alpha < E_p(u(f)|\{s, t\}) \) and \( \alpha M + (1 - \alpha)m \succ_{s} f \) if \( \alpha > E_p(u(f)|\{s, t\}). \) In particular, it follows that \( f \succ_{\{s, t\}} g \) if and only if \( E_p(u(f)|\{s, t\}) > E_p(u(g)|\{s, t\}). \) Since \( f(t) = g(t) \) and \( p(s|\{s, t\}) > 0, \) this is the case if and only if \( E_p(u(f)|\{s, t\}) > E_p(u(g)|\{s, t\}) \) and from above we know that this is the case if and only if \( f \succ_{\{s\}} g. \) Similarly, \( g \succ_{\{s, t\}} f \) if and only if \( g \succ_{\{s\}} f. \)

We claim that “is infinitely less likely than” is a transitive relation. Consider states \( r, s, \) and \( t, \) with \( r \) and \( s \) active, and suppose that \( r \) is infinitely less likely than \( s \) while \( s \) is infinitely less likely than \( t. \) We claim that \( r \) is infinitely less likely than \( t. \) Consider lotteries \( f \) and \( g \) with \( f \succ_{\{r\}} g, f(s) = g(s) \) and \( f(t) = g(t). \) We have \( f \sim_{\{r, s\}} g \) because \( r \) is infinitely less likely than \( s. \) Applying Subjective Substitution gives \( f \sim_{\{r, s, t\}} g, \) and applying Revealed Improbability (with \( R = \{s\}, \) because \( s \) is infinitely less probable than \( t) \) gives \( f \sim_{\{r, t\}} g. \) By Relevance, this is the case even when \( f(s) \neq g(s), \) so \( r \) is indeed
infinitely less likely than \( t \).

Suppose that \( r \) is infinitely less likely than \( s \), and \( t \) is an active state with \( r \) and \( t \) comparably likely. Consider lotteries \( f \) and \( g \) such that \( f \succ_{\{r\}} g \), \( g \succ_{\{t\}} f \), and \( f \sim_{\{r,t\}} g \). Subjective Substitution implies that \( f \sim_{\{r,s,t\}} g \), and Revealed Improbability implies that \( f \sim_{\{s,t\}} g \). Thus \( t \) is infinitely less likely than \( s \).

We claim that there is a complete weak ordering of states \( \preceq \) such that:

(a) \( s \prec t \) whenever \( s \) is infinitely less likely than \( t \).

(b) If \( s \) is inert, then for all \( t \in \Omega \), either \( s \prec t \) or \( t \prec s \).

(c) If \( s \) and \( t \) are active, and neither is infinitely less likely than the other, then \( s \sim t \).

In fact the ordering of active states is completely determined by these conditions. For inert states, we can choose any ordering satisfying (b) such that \( s \prec t \) whenever \( s \) is inert and \( t \) an active state that is not infinitely less likely than \( s \), and \( t \prec s \) whenever \( s \) is inert and \( t \) is infinitely less likely than \( s \).

Now consider a set \( T \in \Xi \). There are two possibilities. The first is that all \( \preceq \)-maximal elements of \( T \) are active. For any \( S \subseteq T \) we set \( p(S|T) = p(S'|T') \) where \( S' \) and \( T' \) are the sets of active elements of \( S \) and \( T \). If \( R \subseteq S \) and \( R' \) is the set of active states of \( R \), then our earlier result gives

\[
p(R|T) = p(R'|T') = p(R'|S')p(S'|T') = p(R|S)p(S|T).
\]

The second possibility is that there is a single \( \preceq \)-maximal \( t \in T \) which is inert. For such a \( T \), if \( S \subseteq T \) we let \( p(S|T) \) be one or zero according to whether \( t \in S \). If \( R \subseteq S \), then \( p(R|T) = p(R|S)p(R|T) \) because both expressions are 1 if \( t \in R \) and they are both 0 otherwise. This concludes the proof that \( p \) is Bayesian.

Consider a lottery \( f \) and a set \( S \). Let \( k_f \) be a lottery such that for each active \( t \in S \), \( k_f(t) \) assigns probability \( E_{f(t)}(u|\{t\}) \) to \( b_t \) and probability \( 1 - E_{f(t)}(u|\{t\}) \) to \( w_t \).

In view of the definition of \( u \), repeated applications of Objective Substitution implies that \( f \sim_{\{t\}} k_f \) for all \( t \), after which Subjective Substitution implies that \( f \sim_S k_f \). Let

\[
E_p(u(f)|S) = \sum_{t \in S} p(t|S)E_{f(t)}(u|\{t\}).
\]
Let $\ell_f$ be $E_p(u(f)|S)$ times $M$ plus $1 - E_p(u(f)|S)$ times $m$. In view of the definition of $p$, repeated applications of Objective Substitution imply that $k_f \sim_S \ell_f$, so $f \sim_S \ell_f$. So long as $S$ has active states, $M \succ_S m$, and Continuity implies that $f \succ_S \alpha M + (1 - \alpha)m$ if $\alpha < E_p(u(f)|S)$ and $\alpha M + (1 - \alpha)m \succ_s f$ if $\alpha > E_p(u(f)|S)$.

C. The theorem is actually if and only if. That is, if the decision maker’s preferences can be characterized by expected utility, then all the axioms are satisfied.

D. Usually there are many different choices of $p$ and $u$ that rationalize the system of preferences $\succeq_S$. In particular, probability and utility are, in a sense, dual, insofar as we can rescale by increasing the utility differences between outcomes in a state while decreasing the probability of that state.