1. Introduction to Differential Equations

1a) At the most general level, a differential equation is an equation containing a derivative of a function. For ordinary differential equations, the function that has been differentiated is a function of one variable. All of our discussion will be restricted to ordinary differential equations.

A differential equation is solved by finding the function that has been differentiated. In the simplest case, the differential equation will have the derivative on the LHS and some expression of the independent variable on the RHS. The general form is:

\[ y'(x) = \phi(x). \]  

(1)

For example:

\[ y'(x) = 2x. \]  

(2)

Such equations can be solved by integrating both sides with respect to \( x \). Integrating the LHS gives, by definition, \( y(x) \). To integrate the RHS, we need to figure out what function, when differentiated with respect to \( x \), will give the expression on the RHS. The most general such expression for our example is \( x^2 + K \), where \( K \) is an arbitrary constant, known as a constant of integration. Thus the solution to the differential equation is

\[ y(x) = x^2 + K. \]  

(3)

Equation (3) defines a whole family of functions, one for each value of \( K \). If we know the value of \( y \) at some value of \( x \), then we can determine the value of \( K \). Suppose that we know that \( y(2) = 6 \). Substituting this into (3) gives

\[ y(2) = 2^2 + K = 6, \]

from which it is plain that \( K = 2 \). Conditions like \( y(2) = 6 \) are known as initial conditions; they provide the means to determine the value of constants that are contained in general solutions of differential equations.
Differential equations like (1) will be familiar. A more difficult type of differential equation involves, not just the derivative and an expression of the independent variable, but the function itself. It has the general form:

\[ y'(x) = \phi(y(x), x). \]

For example:

\[ y'(x) = y(x) \cdot 2x. \] (4)

In this case we cannot simply integrate the RHS because we don’t know what the expression on the RHS is. The RHS includes \( y(x) \) and the formula for \( y(x) \) is precisely what we don’t know and are trying to discover.

There is, however, a trick that we can employ. Assume that \( y(x) > 0 \) for all \( x \). Then we can divide both sides of (4) by \( y(x) \) to yield:

\[ \frac{y'(x)}{y(x)} = 2x. \] (5)

We know from the chain rule that

\[ \frac{d}{dx} \ln y(x) = \frac{y'(x)}{y(x)}. \]

Thus, integrating both sides of (5) with respect to \( x \) gives:

\[ \ln y(x) = x^2 + K. \]

Now \( e \) to the power of the LHS must equal \( e \) to the power of the RHS. Thus:

\[ e^{\ln y(x)} = e^{x^2 + K}, \]

which simplifies to

\[ y(x) = e^{x^2} \cdot e^K. \]

\( e^K \) is a positive constant, which we may write as \( C > 0 \). Thus our solution is

\[ y(x) = Ce^{x^2}. \] (6)

One can confirm by differentiating and substituting, that this last expression does indeed solve (4).

In deriving (6), we assumed that \( y(x) > 0 \) for all \( x \). The cases of \( y(x) < 0 \) for all \( x \) and \( y(x) = 0 \) for all \( x \) can be handled by dropping the restriction that \( C > 0 \) (the formula for those cases can also be derived directly if one has the patience).\(^1\) Which of these three possibilities occurs in any given case is determined from an initial condition, e.g., if \( y(0) = -5 \), then we have

\[ y(0) = Ce^0 = -5, \]

\(^1\) What about the case where \( y(x) \) changes sign? It can’t change sign, as a careful analysis of (4) will show.
so that $C = -5$. Substituting back, we get

$$y(x) = -5e^{x^2} < 0 \text{ for all } x.$$  

The examples above have all involved only a first derivative, but in the general case differential equations may involve derivatives of any order (together with the function itself). The most general expression for an ordinary differential equation is:

$$F\left(y^n(x), y^{n-1}(x), \ldots, y''(x), y'(x), y(x), x\right) = 0,$$

where $y^k(x)$ denotes the $k$th derivative of $y(x)$. This is an $n$th order differential equation: the order of a differential equation equals the order of the highest order derivative that it contains. As always, solving the equation means finding an expression for $y(x)$ that, along with its derivatives, satisfies the equation. With an $n$th order differential equation, there are $n$ initial conditions specifying the value of $y(x)$ and its first $n-1$ derivatives at some initial point.

Solutions to differential equations exist under very general conditions. However, these solutions need not be in terms of familiar elementary functions (such as polynomial, log, exponential and trigonometric functions) even when the differential equation is stated in terms of those functions. Further, even if the solution is in terms of elementary functions, finding the solution may not be easy or obvious. There are some standard tricks for solving certain kinds of non-linear differential equations but, for the most part, soluble equations tend to be linear. As in other areas of calculus, this is not as limiting as it may seem. Linear approximations to non-linear equations can provide valuable information about the “local” behaviour of differential equation solutions.

For the most part, the remainder of the discussion will present mechanical procedures for solving linear equations, without much in the way of derivation.

### 2. Linear Differential Equations: General Principles

2a) For the remainder of these notes, $t$ will be used instead of $x$ as the independent variable. This is of no mathematical consequence but is chosen because the independent variable is often “time” or, more accurately, it represents a date (albeit a date with a zero point that may not correspond to the zero point on any standard calendar). The general form of an $n$th order linear differential equation is:

$$y^n(t) + u_{n-1}(t)y^{n-1}(t) + u_{n-2}(t)y^{n-2}(t) + \cdots + u_1(t)y'(t) + u_0(t)y(t) = w(t), \quad (7)$$

where $y^k(t)$ is the $k$th order derivative of $y(t)$ and the $u_i(t)$ and $w(t)$ are any continuous functions of $t$. The fact that the $u_i(t)$ and $w(t)$ are functions rather than constants may seem a little curious in a “linear” equation; the equation is linear in $y$ and its derivatives for given $t$ — and that is where the terminology originates.

Associated with this equation is the reduced equation formed by replacing $w(t)$ by zero:

$$y^n(t) + u_{n-1}(t)y^{n-1}(t) + u_{n-2}(t)y^{n-2}(t) + \cdots + u_1(t)y'(t) + u_0(t)y(t) = 0. \quad (8)$$
We denote by $y_p$ (the **particular solution**) any solution to (7) — not necessarily the most general — and by $y_c$ (the **complementary function**) the **most general** solution to (8).

We can now state:

**Theorem.** The most general solution to equation (7) is the sum of any solution to (7) and the **most general** solution to (8), i.e.,

$$y(t) = y_p(t) + y_c(t).$$

(9)

The above theorem is simply a generalisation of the proposition that the most general form of solution to an integration problem is the sum of a primitive (or anti-derivative) and a constant: the particular solution corresponds to a primitive and the complementary function corresponds to the constant of integration. The theorem can be proved by considering any two solutions to (7), $y_a(t)$ and $y_b(t)$, and defining the difference between them $d(t) \equiv y_a(t) - y_b(t)$. It is a simple exercise to show that, because $y_a(t)$ and $y_b(t)$ satisfy (7), $d(t)$ must satisfy (8). With the difference between any two solutions satisfying (8), it follows that all possible solutions can be found by finding one solution to (7) and then adding to it the most general solution to (8), which is what the theorem asserts.

**2b)** It follows from the above theorem that the process of solving a linear differential equation may be divided into that of finding any solution to equation (7) — i.e., finding a $y_p$ — and that of finding the most general solution to equation (8) — i.e., finding $y_c$. The assumed continuity of the $u(t)$ and $w(t)$ functions guarantees that $y_c$ and $y_p$ exist — actually finding them is another matter.

In those cases in which we can get a solution for an $n$th order linear differential equation, it will involve $n$ constants. To determine the values of those constants requires $n$ initial conditions of the form

$$y(t_0) = \xi_0, y'(t_0) = \xi_1, \ldots, y^{n-2}(t_0) = \xi_{n-2}, y^{n-1}(t_0) = \xi_{n-1}.$$ 

The initial date $t_0$ is often taken as zero. We will now consider special cases of equation (7).

### 3. First Order Linear Differential Equations

**3a)** These have the general form:

$$y'(t) + u(t)y(t) = w(t)$$

(10)

For these equations $y_c$ is given by:

$$y_c = Ce^{-\int u(t)dt},$$

(11)

where $C$ is a constant (which may be of either sign). One particular solution $y_p$ is:

$$y_p = e^{-\int u(t)dt}\int w(t)e^{\int u(t)dt}dt.$$
Adding these two expressions yields the general solution for a linear first order differential equation:

$$y(t) = e^{-\int u(t) dt} \left( C + \int w(t) e^{\int u(t) dt} dt \right)$$  \hspace{1cm} (13)

The value of $C$ is fixed by the initial condition, $y(t_0) = \xi_0$. Note that, as with inequality constrained optimisation, the validity of the solution formula depends on the differential equation being written in standard form, i.e., in the form of (10).

**3b)** In order to solve for $y(t)$ the following 4 step procedure is recommended.

**STEP 1.** Solve for $\int u(t) dt$.

**STEP 2.** Substitute the result of Step 1 into $\int w(t) e^{\int u(t) dt} dt$ and solve the latter integral.

**STEP 3.** Substitute the results of Steps 1 and 2 into equation (13) — be careful to note the minus sign in the first $e^{-\int u(t) dt}$ and the absence of any minus sign in $\int w(t) e^{\int u(t) dt} dt$.

**STEP 4.** Use the initial condition, $y(t_0) = \xi_0$, to fix the value of the constant $C$. Make sure that this is the last step, only performed after you have added $y_c$ and $y_p$ — you cannot correctly determine the value of $C$ by combining the initial condition with the complementary function $y_c$ alone.

In performing the various integrations, one would normally add constants of integration to the primitives of the functions being integrated. These may be omitted in the above steps since the presence of the $C$ constant renders any additional constants redundant.

4. Linear Differential Equations of Any Order With Constant Coefficients and Term

**4a)** For differential equations of order greater than one, we specialise the analysis by considering only constant $u_i(t)$ coefficients and a constant $w(t)$ term (a technique for dealing with a non-constant $w(t)$ term is described in the Appendix to these notes). These take the general form:

$$y^n(t) + a_{n-1}y^{n-1}(t) + a_{n-2}y^{n-2}(t) + \cdots + a_1y'(t) + a_0y(t) = b,$$  \hspace{1cm} (14)

where the $a_i$ and $b$ are constants. Once again, it is important to remember that, as with inequality constrained optimisation, the validity of the solution formulae depend on the differential equation being written in standard form, i.e., in the form of (14).

**Particular Solutions.**

**4b)** A simple general formula gives the particular solution for all equations of this form. For the purpose of this formula, the coefficient of $y^n(t)$ is taken to be $a_n = 1$ and $y(t)$ is to be regarded as a zero order derivative.

Denote by $y^i(t)$ the lowest order derivative with a non-zero co-efficient. Given the numbering scheme for (14), the coefficient of $y^i(t)$ is $a_i$. A particular solution to (14) is given by:

$$y_p = \frac{b}{a_i i!} t^i.$$  \hspace{1cm} (15)
i! is \(i\) factorial, i.e., \(i \cdot (i-1) \cdot (i-2) \cdots 1\). By definition, \(0! = 1\).

**4c)** The most common special case of this is when \(i = 0\) (i.e., \(y(t)\) has a non-zero coefficient), giving
\[
y_p = \frac{b}{a_0}.
\]
Another special case is where \(i = n\), giving
\[
y_p = \frac{b}{n!} \cdot t^n.
\]

**4d)** Applying (15) to first order equations of the form
\[
y'(t) + ay(t) = b
\]
(where \(a\) represents \(a_0\)), the particular solutions have the following form:
\[
y_p = \begin{cases} 
\frac{b}{a} & \text{if } a \neq 0 \\
bt & \text{if } a = 0.
\end{cases}
\]
This can also be derived as a special case of (12).

**4e)** For second order equations of the form
\[
y''(t) + a_1 y'(t) + a_0 y(t) = b,
\]
application of (15) yields the particular solution:
\[
y_p = \begin{cases} 
\frac{b}{a_0} & \text{if } a_0 \neq 0 \\
\frac{(bt)/a_1}{a_0} & \text{if } a_0 = 0, a_1 \neq 0 \\
\frac{(bt^2)/2}{a_0} & \text{if } a_0 = 0, a_1 = 0
\end{cases}
\]

**Complementary Functions.**

**4f)** It is not so straightforward to state a formula for the complementary function \(y_c\) in this case as it was for first order linear equations. As a first step, form the **characteristic equation**:
\[
r^n + a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \cdots + a_1 r + a_0 = 0.
\]
This equation is derived from equation (14) by first replacing \(b\) by zero — thus forming the **reduced equation** encountered in general form earlier in these notes as equation (8) — and then replacing the \(k\)th order derivative by \(r^k\) (for the purpose of this substitution, \(y(t)\) itself is regarded as a zero order derivative and is therefore replaced by \(r^0 = 1\)).

**4g)** Equation (17) is a \(n\)th degree polynomial and will have \(n\) roots, which may be distinct or repeated and real or complex.\(^2\)

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\(^2\) If you are wondering what it means for a root to be repeated, then the answer is as follows. Any polynomial function of the form: \(P(r) = r^n + a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \cdots + a_1 r + a_0\) may be factorised into the form: \(P(r) = (r - r_1)(r - r_2) \cdots (r - r_n)\), where the \(r_i\) are the solutions/roots of the polynomial equation. If two \(r_i\) values are equal, then the \(r_i\) root is repeated, e.g., if \(P(r) = r^2 - 14r + 49\), then this can be factorised into \(P(r) = (r - 7)(r - 7)\), implying that 7 is a repeated root.
Complex roots always occur in conjugate pairs. From these roots, we may form the complementary function as follows:

(i) Each distinct real root \( r_j \) contributes an expression \( C_j e^{r_j t} \) to \( y_c \).

(ii) If a real root is repeated \( k \) times, i.e.,

\[
r_j = r_{j+1} = \cdots = r_{j+(k-1)} = \bar{r},
\]

then it contributes:

\[
C_j e^{rt} + C_{j+1} e^{r(t+1)} + \cdots + C_{j+(k-1)} e^{r(t+k-1)} = \sum_{s=0}^{k-1} C_{j+s} e^{r(t+s)}.
\]

(18)

to \( y_c \).

(iii) Each pair of complex conjugate roots \( r_j, r_{j+1} \) in the form \( \alpha \pm vi \) contributes:

\[
B_j e^{\alpha t} + B_{j+1} e^{\alpha(t+1)} + \cdots + B_{j+(k-1)} e^{\alpha(t+k-1)} = \sum_{s=0}^{k-1} B_{j+s} e^{\alpha(t+s)}
\]

to \( y_c \), which is more conveniently transformed into:

\[
e^{\alpha t} [C_j \cos(\beta t) + C_{j+1} \sin(\beta t)]
\]

(19)

4h) As a special case, the first order equation

\[
y'(t) + ay(t) = b
\]
yields a characteristic equation of

\[
r + a = 0,
\]

with the real root, \( r = -a \). Thus \( y_c = Ce^{-at} \). This could also be derived from (11).

4i) As another special case, the second order equation

\[
y''(t) + a_1 y'(t) + a_0 y(t) = b
\]
yields a characteristic equation:

\[
r^2 + a_1 r + a_0 = 0.
\]

This has solutions given by the quadratic formula:

\[
r = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_0}}{2}
\]

Depending on the solution values of \( r \), \( y_c \) takes one of three forms.

---

\(^3\) See Simon & Blume Appendix A3 for the mathematics underlying this transformation. If there are repeated pairs of complex conjugate roots, then a formula similar to (18) is appropriate: if a complex pair is repeated \( k \) times, then it will contribute \( \sum_{s=0}^{k-1} e^{r(t+s)} [C_{j+s} \cos(\beta t) + C_{j+s+1} \sin(\beta t)] \) to \( y_c \).

\(^4\) In any formula relating to differential/difference equations, the angle \( \theta \) in \( \cos \theta \) and \( \sin \theta \) is expressed in radians. 1 radian is approximately 57\(^\circ\); 2\(\pi\) radians equal exactly 360\(^\circ\). Further, \( d/d\theta(\cos \theta) = -\sin \theta \) and \( d/d\theta(\sin \theta) = \cos \theta \).
1) \((a_1)^2 - 4a_0 > 0 \implies\) two real roots, \(r_1\) and \(r_2\), with
\[ y_c = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]
2) \((a_1)^2 - 4a_0 = 0 \implies\) one repeated real root, \(r\), with
\[ y_c = C_1 e^{rt} + C_2 t e^{rt}. \]
3) \((a_1)^2 - 4a_0 < 0 \implies\) two complex roots, \(r_1\) and \(r_2\) in the form \(\alpha \pm \beta i\), with
\[ y_c = e^{\alpha t} [C_1 \cos \beta t + C_2 \sin \beta t]. \]

5. **Summary of Procedure for Solving Linear Differential Equations With Constant Coefficients and Terms**

**STEP 1.** Solve the characteristic equation:
\[ r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \cdots + a_1 r + a_0 = 0. \]

**STEP 2.** Substitute the solution values of \(r\) into the appropriate expressions for \(y_c\) given in 4g).

**STEP 3.** Solve for a particular solution.

**STEP 4.** Add the results of Steps 2 and 3 to get a general expression for \(y(t)\).

**STEP 5.** Use the initial conditions,
\[ y(t_0) = \xi_0, y'(t_0) = \xi_1, \ldots, y^{n-2}(t_0) = \xi_{n-2}, y^{n-1}(t_0) = \xi_{n-1}, \]
to fix the \(C\) constants appearing in the general expression for \(y(t)\). **Make sure you differentiate** \(y(t)\) \(k\) times before attempting to use \(y^k(t_0) = \xi_k\) and, in differentiating sines and cosines, **do not forget to use the chain rule**, e.g.,
\[ \frac{d}{dt} \cos \beta t = (-\sin \beta t) \beta = -\sin \beta t. \]

Once again, it is important that fixing the values of the \(C\) constants is the last step, performed only after \(y_c\) and \(y_p\) have been added.

6. **Stability of solutions to constant coefficient linear differential equations**

6a) In many cases, the particular solution has an interpretation as an economic equilibrium. We therefore say that the equilibrium is stable if \(y(t) \to y_p\) as \(t \to \infty\). Plainly, this condition will be satisfied if and only if \(y_c \to 0\) as \(t \to \infty\).

From a study of the formulae for the complementary function, we get the following result: \(y_c \to 0\) as \(t \to \infty\) for arbitrary initial conditions (and hence arbitrary \(C\) constants) if and only if the **real part of each** root of the characteristic equation is negative, i.e., \(r_j < 0\) when \(r_j\) is real; and \(\alpha < 0\) when there are complex roots. This is obvious from inspection in the case of distinct real roots. With repeated real roots, it is less obvious because there
are offsetting influences in the expression $t^s e^{rt}$ for $s \geq 1$. It turns out, however, that the $e^{rt}$ expression always "wins" if $\bar{r}$ is negative, so that the expression goes to zero. In the case of complex roots, the expression in square brackets in

$$e^{\alpha t} [C_j \cos(\beta t) + C_{j+1} \sin(\beta t)]$$

follows a path of constant oscillation. Whether the oscillation of the expression as a whole is of increasing, constant or diminishing amplitude depends on whether $\alpha$ in the expression $e^{\alpha t}$ is positive, zero or negative, respectively.

A special case is the so-called "saddle point" case in which there are both positive and negative real roots: Such equations are unstable unless the initial conditions are such that the $C$ constants corresponding to the positive roots turn out to be zero.

6b) In the first order case, $r = -\alpha$. Thus $r < 0$, and hence the solution is stable (for arbitrary initial conditions), iff $\alpha > 0$. Less obviously, the condition for stability in the second order case is: $a_1, a_0 > 0$. For higher order differential equations, stability tests take a less simple form.

7. Linear Approximations and Graphical Methods

For those first order non-linear differential equations for which an explicit solution cannot be found, there are two alternative techniques that can be used to gain additional information. Both apply to what are called autonomous equations. These take the form:

$$y'(t) = f(y(t))$$

which is a special case of the more general form of

$$y'(t) = \phi(y(t), t).$$

Equation (20) is said to be autonomous because $y'(t)$ depends only on $y$ and not (directly) on $t$; it is thus "autonomous" with respect to $t$. Equation (20) is the generalisation of the first order linear equation with constant coefficient and term.

The analysis proceeds as follows. Suppose that there exists some value of $y$, call it $y^*$, such that

$$f(y^*) = 0.$$

This is known as a stationary solution to the differential equation (since it implies that $y'(t) = 0$ and hence that the value of $y$ will stay at $y^*$). The differential equation is said to be stable if $y \to y^*$ as $t \to \infty$.$^5$

Take a linear approximation of $f$ at the point $y = y^*$ (geometrically, this is the tangent line to $f$ at $y = y^*$). Substituting this linear approximation into (20) in place of $f$ gives:

$$y'(t) = f(y^*) + f'(y^*) \cdot (y - y^*).$$

$^5$ There are a number of different types of stability identified in the literature, but the definition given will suffice for our purposes.
Noting that $f(y^*) = 0$, this may be rearranged into:

$$y'(t) - f'(y^*) \cdot y = -f'(y^*) \cdot y^*. \quad (21)$$

This is in the form of

$$y'(t) + ay(t) = b \quad (22)$$

with $a = -f'(y^*)$ and $b = -f'(y^*) \cdot y^*$. As noted earlier, (22) is stable iff $a > 0$. Thus (21) is stable iff $f'(y^*) < 0$. We now may state the linear approximation theorem:

Assume that $f$ is a $C^1$ function and that $f(y^*) = 0$. The differential equation (20) is stable on some interval about $y^*$ if (but not only if) the linear approximation of (20) about $y^*$ is stable, i.e., if $f'(y^*) < 0$. It is unstable (globally) if (but not only if) $f'(y^*) > 0$. If $f'(y^*) = 0$, then further information is required.

Note that if $f(y^*) = 0$ and $f'(y) < 0$ for all $y$, then the differential equation is globally stable.

A second approach to non-linear equations of the form of (20) is a graphical one. This involves graphing $f(y)$ on the vertical axis against $y$ on the horizontal axis. If $f(y) > 0$, then $y'(t) > 0$ and $y$ is increasing; if $f(y) < 0$, then $y'(t) < 0$ and $y$ is decreasing. Where $f(y) = 0$ (i.e., the graph of $f$ cuts the horizontal axis), we have a stationary solution. From the previous results, we see that (assuming $f$ is $C^1$) we have a locally stable equilibrium if $f(y) = 0$ and $f'(y) < 0$, and a locally unstable equilibrium if $f(y) = 0$ and $f'(y) > 0$. Given sufficient information about $f$, one can deduce the path of $y$ from any given starting point. See Simon & Blume (pp. 666–669) for examples.

8. Introduction to Difference Equations

A difference equation is an equation containing values of a function at different integer values in its domain (the domain itself consists only of integers). A simple example is

$$y_{t+1} = 3 \cdot y_t. \quad (23)$$

t and $t + 1$ represent two different integers in the domain of the $y$ function, one unit apart. $y_t$ and $y_{t+1}$ are simply the values of the $y$ function at points $t$ and $t + 1$, respectively; it is another way of writing $y(t)$ and $y(t + 1)$ and is a style of notation commonly used when a function has only integers in its domain. Typically these values represent successive dates (days, months, years or whatever) but, from a purely mathematical point of view, $t$ could represent any independent integer variable.

Different values of $t$ give rise to different specialisations of (23). Thus:

$$t = 0 \implies y_1 = 3 \cdot y_0$$

$$t = 1 \implies y_2 = 3 \cdot y_1$$

$$t = 2 \implies y_3 = 3 \cdot y_2$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$t = n \implies y_{n+1} = 3 \cdot y_n$$
The differential equation is solved by finding a $y$ function that satisfies the equation for every value of $t \geq t_0$, where $t_0$ is typically equal to zero, but may be a positive number.

As with differential equations, an initial condition is required to fix the solution function, e.g., $y_0 = 2$.

One way to solve (23) is to substitute the initial condition into the first equation in (24) and thereby solve for $y_1$. The value for $y_1$ can then be substituted into the second equation to solve for $y_2$, which can be used to solve for $y_3$ and so on. This process, however, is not practical if one wishes to find the value of $y_t$ for large $t$. The form of the general solution is clear, however. With each successive value being three times the size of the preceding value, the general solution must be:

$$y_t = 3^t \cdot y_0,$$

where $y_0$ is given by the initial condition.

The most general form of a difference equation is

$$F(y_{t+n}, y_{t+n-1}, \ldots, y_{t+1}, y_t, t) = 0. \quad (25)$$

$t$, $t+1$, $t+2$ etc. represent different integers in the domain of the $y$ function, one unit apart.

The order of a difference equation is the maximum difference between the domain values at which $y$ is evaluated. (25) is thus an $n$th order difference equation since $(t+n) - t = n$. As with an $n$th order differential equation, an $n$th order difference equation requires $n$ initial conditions, giving the value of $y$ at $n$ successive values in its domain (each one unit apart). Note, however, the contrast with differential equations in that $n$ initial conditions for a differential equation involve a specification of the value of the function and its first $n-1$ derivatives at a single point in its domain.

Solutions to difference equations exist under very general conditions — even more general than for differential equations owing to the possibility of solving them by a generalisation of the method described in connection with equations (24). But, once again, finding solutions expressible in terms of familiar formulae usually involves working with linear equations.

The remainder of the discussion will present mechanical procedures for solving linear equations, without much in the way of derivation.

9. Linear Difference Equations: General Principles

9a) The analysis of linear difference equations has many similarities with that of linear differential equations. We will only consider the constant coefficient case. This has the general form:

$$y_{t+n} + a_{n-1}y_{t+(n-1)} + a_{n-2}y_{t+(n-2)} + \cdots + a_1y_{t+1} + a_0y_t = w_t, \quad (26)$$

where the $a_i$ are constants. Associated with this equation is the reduced equation formed by replacing $w_t$ by zero:

$$y_{t+n} + a_{n-1}y_{t+(n-1)} + a_{n-2}y_{t+(n-2)} + \cdots + a_1y_{t+1} + a_0y_t = 0. \quad (27)$$
We denote by \( y_p \) (the **particular solution**) any solution to (26) — not necessarily the most general — and by \( y_c \) (the **complementary function**) the most general solution to (27). We can now state:

**Theorem.** The most general solution to equation (26) is the sum of any solution to (26) and the most general solution to (27), i.e.,

\[
y_t = y_{p,t} + y_{c,t}.
\]  
(28)

The analogy with differential equations should be clear.

9b) It follows from the above theorem that, just as with differential equations, the process of solving a linear difference equation can be divided into that of finding any solution to equation (26) — i.e., finding a \( y_p \) — and that of finding the most general solution to equation (27) — i.e., finding \( y_c \).

In those cases in which we can get a explicit solution for an \( n \)th order linear difference equation, it will involve \( n \) constants. To determine the values of those constants requires \( n \) initial conditions of the form

\[
y_0 = \xi_0, y_1 = \xi_1, \ldots, y_{n-2} = \xi_{n-2}, y_{n-1} = \xi_{n-1}.
\]

The dates do not have to start at zero (they could go, say, from 5 to 5 + (\( n - 1 \)) ), but usually they do start at zero.

10. **First Order Linear Difference Equations With Constant Coefficient and Term**

10a) These have the general form:

\[
y_{t+1} + ay_t = b
\]  
(29)

For these equations \( y_c \) is given by:

\[
y_c = C(\lambda)^t \quad \text{where } \lambda = -a
\]  
(30)

(\( C \) is a constant of either sign). One particular solution \( y_p \) is:

\[
y_{p,t} = \begin{cases} 
\frac{b}{1 + a} & \text{if } a \neq -1 \\
bt & \text{if } a = -1 
\end{cases}
\]  
(31)

The overall solution is the sum of the particular solution and the complementary function:

\[
y_t = \begin{cases} 
\frac{b}{1 + a} + C(-a)^t & \text{if } a \neq -1 \\
bt + C(-a)^t & \text{if } a = -1 
\end{cases}
\]  
(32)

The value of \( C \) is fixed by use of an initial condition \( y_0 = \xi_0 \). See the Appendix to these notes for a discussion of the case where the constant term \( b \) is replaced by the variable term \( w_t \).
11. Linear Difference Equations of Any Order With Constant Coefficients and Term

11a) Recall the general form given in (26), which we specialise to the case of a constant term $b$:

$$y_{t+n} + a_{n-1}y_{t+(n-1)} + a_{n-2}y_{t+(n-2)} + \cdots + a_1y_{t+1} + a_0y_t = b,$$  \hspace{1cm} (33)

Particular Solutions.

11b) One particular solution is

$$y_p = \frac{b}{1 + a_n + a_{n-1} + \cdots + a_0} = \frac{b}{\sum_{i=0}^{n} a_i}$$  \hspace{1cm} assuming $\sum_{i=0}^{n} a_i \neq 0$,

where the coefficient of $y_{t+n}$ is taken to be $a_n = 1$.

If it should be the case that $\sum_{i=0}^{n} a_i = 0$, then, unlike in the differential equations case, there is no neat formula for $y_p$ (that I know of). Instead, the procedure is to try $y_{p,t} = kt$, substitute it into (33) and solve for $k$. If this also involves division by zero, try $y_{p,t} = kt^2$ and so on.

11c) For first order equations, $y_{p,t} = kt$ is as far as you will ever need to go. Particular solutions for every possibility have already been given as (31).

11d) For second order equations of the form

$$y_{t+2} + a_1y_{t+1} + a_0y_t = b,$$  \hspace{1cm} (34)

all possible cases are covered by the following formula for $y_{p,t}$:

$$y_{p,t} = \begin{cases} 
\frac{b}{1 + a_1 + a_0} & \text{if } 1 + a_1 + a_0 \neq 0 \\
\frac{(bt)}{(a_1 + 2)} & \text{if } 1 + a_1 + a_0 = 0, a_1 + 2 \neq 0 \\
\frac{(bt^2)}{2} & \text{if } 1 + a_1 + a_0 = 0, a_1 + 2 = 0.
\end{cases}$$

Complementary Functions.

11e) To derive the complementary function $y_c$, form the characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0.$$  \hspace{1cm} (35)

This equation is derived from equation (33) by first replacing $b$ by zero — thus forming the reduced equation encountered earlier in these notes as equation (27) — and then replacing the $y_{t+k}$ by $\lambda^k$. 
11f) Equation (35) is an $n$th degree polynomial and will have $n$ roots, which may be distinct or repeated and real or complex. Complex roots always occur in conjugate pairs. From these roots, we may form the complementary function as follows:

(i) Each distinct real root $\lambda_j$ contributes an expression $C_j \lambda_j^t$ to $y_c$.

(ii) If a real root is repeated $k$ times, i.e.,

$$\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+(k-1)} = \bar{\lambda},$$

then it contributes:

$$C_j(\bar{\lambda})^t + C_{j+1}t(\bar{\lambda})^t + \cdots + C_{j+(k-1)t^{k-1}(\bar{\lambda})^t} = \sum_{s=0}^{k-1} C_{j+s} t^s(\bar{\lambda})^t. \quad (36)$$

to $y_c$.

(iii) Each pair of complex conjugate roots $\lambda_j, \lambda_{j+1}$ in the form $\alpha \pm vi$ contributes:

$$B_j(\alpha + vi)^t + B_{j+1}(\alpha - vi)^t$$

to $y_c$, which is more conveniently transformed into.\(^6\)

$$R^t [C_j \cos(\theta t) + C_{j+1} \sin(\theta t)] \quad (37)$$

where

$$R = \sqrt{\alpha^2 + \beta^2}$$

and

$$\cos \theta = \frac{\alpha}{R}, \sin \theta = \frac{\beta}{R} \quad \text{or, equivalently,} \quad \theta = \cos^{-1} \frac{\alpha}{R} = \sin^{-1} \frac{\beta}{R}$$

11g) As a special case, the first order equation

$$y_{t+1} + ay_t = b$$

yields a characteristic equation of

$$\lambda + a = 0,$$

with the real root, $\lambda = -a$. Thus $y_c = C(-a)^t$. This was given earlier.

11h) The second order difference equation:

$$y_{t+2} + a_1 y_{t+1} + a_0 y_t = b$$

yields a characteristic equation:

$$\lambda^2 + a_1 \lambda + a_0 = 0.$$
This has solutions given by the quadratic formula:

\[ \lambda = \frac{-a_1 \pm \sqrt{(a_1)^2 - 4a_0}}{2} \]

Depending on the solution values of \( \lambda \), \( y_c \) takes one of three forms.

1) \((a_1)^2 - 4a_0 > 0 \implies\) two real roots, \( \lambda_1 \) and \( \lambda_2 \), with

\[ y_c = C_1(\lambda_1)^t + C_2(\lambda_2)^t \]

2) \((a_1)^2 - 4a_0 = 0 \implies\) one repeated real root, \( \lambda \), with

\[ y_c = C_1(\lambda)^t + C_2t(\lambda)^t \]

3) \((a_1)^2 - 4a_0 < 0 \implies\) two complex roots, \( \lambda_1 \) and \( \lambda_2 \) in the form \( \alpha \pm vi \), with

\[ y_c = R^t \left[ C_1 \cos \theta t + C_2 \sin \theta t \right] . \]

where

\[ R = \sqrt{\alpha^2 + \beta^2} \]

and

\[ \cos \theta = \frac{\alpha}{R}, \sin \theta = \frac{\beta}{R} \]

or, equivalently,

\[ \theta = \cos^{-1} \frac{\alpha}{R} = \sin^{-1} \frac{\beta}{R} \]

12. Summary of Procedure for Solving Linear Difference Equations With Constant Coefficients and Terms

STEP 1. Solve the characteristic equation:

\[ \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0. \]

STEP 2. Substitute the solution values of \( \lambda \) into the appropriate expressions for \( y_c \) given in 11f).

STEP 3. Solve for a particular solution.

STEP 4. Add the results of Steps 2 and 3 to get a general expression for \( y_t \).

STEP 5. Use the initial conditions,

\[ y_0 = \xi_0, y_1 = \xi_1, \ldots, y_{n-2} = \xi_{n-2}, y_{n-1} = \xi_{n-1} \]

to fix the values of the \( C \) constants appearing in the general expression for \( y_t \). **Once again, it is important that this be the last step, performed only after \( y_c \) and \( y_p \) have been added.**
13. Stability of solutions to constant coefficient linear difference equations

13a) As with differential equations, we must find the conditions for \( y_c \to 0 \) as \( t \to \infty \). Regardless of the nature of the roots, the condition for stability is the same: the absolute value of all roots must be less than one. Note in this connection that, by definition:

\[
|\alpha + vi| = |\alpha - vi| = \sqrt{\alpha^2 + \beta^2} = R
\]

The preceding condition for difference equations may be contrasted with the corresponding condition for differential equations: that the real parts of all roots must be negative. The sign of real roots does, however, have a significance for difference equations. If \( \lambda < 0 \), then \( \lambda^t \) oscillates, whereas for \( \lambda \geq 0 \), \( \lambda^t \) is a monotonic function of \( t \).

As with differential equations, there is a “saddlepoint” case: in the difference equations case, this is where there is one real root with an absolute value less than 1 and one real root with an absolute value greater than 1. This case is stable only if the \( C \) constant corresponding to the root with absolute value greater than 1 happens to be zero.

13b) In the first order case, \( \lambda = -a \). Thus \( |\lambda| < 1 \), and hence the solution is stable, iff \( |a| < 1 \). Less obviously, the following conditions, taken together, are necessary and sufficient for stability in the second order case:

\[
a_0 < 1 \\
|a_1| < 1 + a_0.
\]

14. Graphical Methods

As with differential equations, one can use graphical methods to analyse non-linear first order autonomous difference equations. Linear approximations are less useful because the notion of “local” stability is not well defined since changes in \( t \) must always be in integer steps, leading to discrete (rather than continuous) changes in \( y_t \).

A first order autonomous difference equation takes the form of

\[ y_{t+1} = f(y_t). \]

A graphical analysis involves graphing \( f(y_t) \) on the vertical axis and \( y_t \) on the horizontal axis. Stationary equilibria occur where the graph crosses the 45° line, since at this point \( y_{t+1} = y_t \), and this situation is perpetuated indefinitely. A sufficient condition for a stationary equilibrium to be stable is that the slope of \( f \) have an absolute value less than 1 for all \( y_t \). The path of \( y_t \) will be monotonic if \( f \) has a positive slope and oscillating if \( f \) has a negative slope.

Given a starting value of \( y_t \), its future path can be shown graphically, as illustrated in Figure 1.
$y_{t+1} = y_t + 1$

Figure 1
Differential Equations.

Following is a description of the method of undetermined coefficients for finding the particular solution for differential equations with a non-constant term, i.e., for differential equations of the form:

\[ y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = w(t). \]  

(1A)

It is applicable if the variable term \( w(t) \) and its successive derivatives contain only a finite number of distinct types of expressions (not counting as distinct types of expressions those that are merely constant multiples of other expressions).

For example, if \( w(t) = t^3 \), then we have

\[
\begin{align*}
  w'(t) &= 3t^2 \\
  w''(t) &= 6t \\
  w'''(t) &= 6 \\
  w^k(t) &= 0 \text{ for all } k > 3
\end{align*}
\]

Thus there are four expression types: \( t^3, t^2, t \) and a constant \( K \).

On the other hand, if \( w(t) = t^{-1} \), then

\[
\begin{align*}
  w'(t) &= -t^{-2} \\
  w''(t) &= 2t^{-3} \\
  w'''(t) &= -6t^{-4} \\
  w^k(t) &= (-1)^k k! t^{-(k+1)} \text{ for all } k \geq 1
\end{align*}
\]

Thus each successive derivative involves a different power of \( t \), giving rise to an infinite number of distinct expression types.

If there are only a finite number of distinct types of expressions, then the procedure is as follows:

STEP 1. Identify all distinct expression types involved in \( w(t) \) and its successive derivatives. Denote these by \( f_1(t), f_2(t), \ldots, f_m(t) \)

If \( w(t) \) and/or any of its derivatives involves a (non-zero) constant term, then a constant should be included among the \( f \) functions.

STEP 2. Let

\[ y_p = B_1 f_1(t) + B_2 f_2(t) + \cdots + B_m f_m(t), \]

(2A)

where the \( B_1, \ldots, B_m \) are constants to be determined (the “undetermined coefficients”).

STEP 3. Substitute \( y_p \) and its derivatives into the differential equation (1A) and equate coefficients on the left- and right-hand sides, e.g., if \( t^2 \) appears on both sides, then equate
the coefficient of $t^2$ on the LHS with the coefficient of $t^2$ on the RHS (if an expression type appears on only the LHS, then its coefficient must be equated to zero — if it only appears on the RHS, then the procedure has failed). This will produce a set of linear equations which can be solved for $B_1, \ldots, B_m$. Substituting these back into $(2A)$ yields $y_p$.

There are two circumstances in which the preceding procedure will not work (even given that $w(t)$ and its derivatives involve only a finite number of distinct expression types). The first is where $w(t)$ is a polynomial function of $t$ and $y(t)$ has a zero coefficient. By construction, $y_p$ in $(2A)$ includes every possible expression type in $w(t)$ but, if $w(t)$ and hence $y_p$ in $(2A)$ is a polynomial, then all of the derivatives of $y_p$ will be polynomials of a lesser degree than $y_p$ and hence polynomials of a lesser degree than $w(t)$. Accordingly, if $y(t)$ has a zero coefficient, then there will appear expression types on the RHS of equation $(1A)$ which will not appear on the LHS of $(1A)$ when $y_p$ and its derivatives are substituted into it; consequently, $y_p$ will not solve the differential equation. A way around this problem, if it appears, is to multiply the RHS of $(2A)$ by $t$:

$$y_p = t (B_1 f_1(t) + B_2 f_2(t) + \cdots + B_m f_m(t))$$

and then proceed as before. This works because, as an application of the product rule,

$$y'_p = B_1 f_1(t) + B_2 f_2(t) + \cdots + B_m f_m(t) + t \left( B_1 f'_1(t) + B_2 f'_2(t) + \cdots + B_m f'_m(t) \right)$$

so $y'_p$ includes all the expression types in $w(t)$.

If both $y(t)$ and $y'(t)$ have zero coefficients, then multiply by $t^2$ instead of $t$ . . . and so on.

The second circumstance in which the preceding procedure does not work is when $w(t)$ and its derivatives happen to solve the reduced equation. This has the result that substituting $y_p$ into the differential equation makes the LHS of that equation identically zero so that it is impossible to equate the LHS and the RHS. The same solution as suggested above also works in this case, i.e., multiply the RHS of $(2A)$ by $t$:

$$y_p = t (B_1 f_1(t) + B_2 f_2(t) + \cdots + B_m f_m(t))$$

and then proceed as before. If that doesn’t work, multiply by $t^2$.

**Difference Equations.**

As with differential equations, the method of undetermined coefficients can be applied to difference equations with a non-constant term, i.e., those of the form:

$$y_{t+n} + a_{n-1}y_{t+(n-1)} + a_{n-2}y_{t+(n-2)} + \cdots + a_1 y_{t+1} + a_0 y_t = w_t. \quad (3A)$$

The method is applicable with difference equations if $w_t$, $w_{t+1}$, $w_{t+2}$ and so on contain only a finite number of distinct types of expressions (not counting as distinct types of expressions those that are merely constant multiples of other expressions).

Note in this regard that an expression like $w_t = K^t$ qualifies (where $K$ is a constant), because $K^t$, $K^{t+1}$, $K^{t+2}$ etc. are constant multiples of each other. More precisely, the ratio
between $K^t$ and $K^{t+n}$ ($n \geq 1$) equals $K^n$ and is thus independent of $t$. This means that there is only one expression type, of the form $K^t$. Following this reasoning, it might seem that $w_t = t^2$ would fail because the ratio between $t^2$ and $(t + n)^2$ depends on $t$. However, $(t + n)^2 = t^2 + 2nt + n^2$, so if we make our expression types $t^2$, $t$ and the constant $K$, then we see that the expressions in $w_t$ and $w_{t+n}$ differ by ratios that are independent of $t$.

An expression that does fail the test is $w_t = t^t$. The ratio between $t^t$ and $(t + n)^{t+n}$ depends on $t$ and any attempt to expand out $(t + n)^{t+n}$ to show that it is composed of a qualifying set of expression types (as was done with $(t + n)^2$ in the previous paragraph) fails because the number of expression types required is not finite but instead increases with $t$.

If $w_t, w_{t+1}$ and so on contain only a finite number of distinct types of expressions, then the procedure is as follows:

**STEP 1.** Identify all distinct expression types involved in $w_t, w_{t+1}$ etc. Denote these by $f_1^t, f_2^t, \ldots, f_m^t$

If $w_t, w_{t+1}$ etc. include a (non-zero) constant term, then a constant should be included among the $f$ functions.

**STEP 2.** Let

$$y_p = B_1 f_1^t + B_2 f_2^t + \cdots + B_m f_m^t,$$

(4A)

where the $B_1, \ldots, B_m$ are constants to be determined (the “undetermined coefficients”).

**STEP 3.** Substitute $y_p$ into the difference equation (3A) and equate coefficients on the left- and right-hand sides. This will produce a set of linear equations which can be solved for $B_1, \ldots, B_m$. Substituting these back into (4A) yields $y_p$.

As with differential equations, it is possible that the preceding procedure will not work (even given that $w_t, w_{t+1}$ etc. involve only a finite number of distinct expression types). Again, as with differential equations, the way to proceed is to multiply (4A) by $t$:

$$y_p = t \left( B_1 f_1^t + B_2 f_2^t + \cdots + B_m f_m^t \right)$$

and then proceed as before. If this should still fail, multiply by $t^2$. 