1. Quadratic Forms

1a) A **quadratic form** is a function $Q: \mathbb{R}^n \rightarrow \mathbb{R}$ with values given by:

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

or, in matrix notation,

$$Q(x_1, \ldots, x_n) = \mathbf{x}' \mathbf{A} \mathbf{x},$$

where $\mathbf{A}$ is an $n$th order square matrix, $\mathbf{x}$ is a column vector and $\mathbf{x}'$ (the transpose of $\mathbf{x}$) is a row vector. In other words, each element $a_{ij}$ of a square matrix is multiplied by $x_i x_j$ and the products are then added.

The quadratic form can be written as the sum of $n$ terms of the form

$$a_{ii}(x_i)^2$$

and $(n^2 - n)/2$ terms of the form

$$a_{ij}x_i x_j + a_{ji}x_j x_i = (a_{ij} + a_{ji})x_i x_j,$$

where $i \neq j$. It will therefore make no difference to the value of the quadratic form if we take the average of $a_{ij}$ and $a_{ji}$ and treat each of $a_{ij}$ and $a_{ji}$ as equal to this average, i.e., if we transform $\mathbf{A}$ into a symmetric matrix by averaging component pairs $a_{ij}$ and $a_{ji}$. Hereafter $\mathbf{A}$ is assumed to be symmetric.

1b) The quadratic form is said to be

(i) **positive definite** if $\mathbf{x}' \mathbf{A} \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$,

(ii) **negative definite** if $\mathbf{x}' \mathbf{A} \mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$,

(iii) **positive semi-definite** if $\mathbf{x}' \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x}$, and

(iv) **negative semi-definite** if $\mathbf{x}' \mathbf{A} \mathbf{x} \leq 0$ for all $\mathbf{x}$.

(v) **indefinite** if $\mathbf{x}' \mathbf{A} \mathbf{x}$ takes on both positive and negative values for suitably chosen $\mathbf{x}$.

Note that if a quadratic form is positive definite, we say that the associated $\mathbf{A}$ matrix is positive definite (i.e., positive definiteness of $\mathbf{x}' \mathbf{A} \mathbf{x}$ and positive definiteness of $\mathbf{A}$ mean the same thing), and similarly for other types of definiteness.
Determinant Tests of Definiteness of Quadratic Forms

2a) We can test for the definiteness of a quadratic form using determinants. First we need a definition. A $k$th order principal submatrix of an $n$th order matrix $A$ is a submatrix formed by deleting $n - k$ rows from $A$ and the same columns from $A$ (e.g., one might delete rows 1, 2 and 5 and columns 1, 2 and 5). The determinant of a $k$th order principal submatrix is called a $k$th order principal minor.

The $k$th order leading principal minor is the determinant of the $k$th order principal submatrix formed by deleting the last $n - k$ rows and columns.

A matrix is positive definite iff all its $k$th order leading principal minors are positive, i.e.,

$$|a_{11}| > 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad \ldots \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0$$

A matrix is negative definite iff all its $k$th order leading principal minors alternate in sign, starting from negative (equivalently, if the minors of odd-numbered order are negative and those of even-numbered order are positive):

$$|a_{11}| < 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0 \quad \ldots$$

2b) For semi-definiteness, we replace the strict inequalities in the above statements with weak inequalities and, most importantly, the weak inequalities must hold for all $k$th order principal minors, not just the leading principal minors.

2c) A very simple necessary condition for the quadratic form to be positive definite is that $a_{ii} > 0$ for all $i$. For positive semi-definiteness, the condition is that $a_{ii} \geq 0$ for all $i$. For negative definiteness, it is necessary that $a_{ii} < 0$ for all $i$. For negative semi-definiteness, it is necessary that $a_{ii} \leq 0$ for all $i$.

2d) Sometimes in economics (particularly in the context of optimisation problems) we are interested in the sign definiteness of a quadratic form for $x$ values satisfying some equality constraint. Formally, we are interested in the sign definiteness of $x'Ax$ for $x$ satisfying $Bx = 0$, where $B$ is an $m \times n$ matrix and, importantly, $m < n$.

To test this, we may form the following bordered matrix:

$$K = \begin{bmatrix} 0 & B \\ B' & A \end{bmatrix}$$

$x'Ax$ is positive definite for $x \neq 0$ satisfying $Bx = 0$ iff the last $n - m$ leading principal minors of $K$ have the same sign as $(-1)^m$ (where $m$ is the number of rows in $B$). $x'Ax$ is
negative definite for \( x \neq 0 \) satisfying \( Bx = 0 \) iff the last \( n-m \) leading principal minors of \( K \) alternate in sign, with \( K \) itself (the last leading principal minor) having the same sign as \((-1)^n\) (where \( n \) is the number of rows/columns in \( A \)).

Tests for semi-definiteness are complicated and will not be discussed here. \( x'Ax \) is indefinite for \( x \neq 0 \) satisfying \( Bx = 0 \) if both the test for positive definiteness and the test for negative definiteness are violated by a non-zero leading principal minor. If they are violated by a zero leading principal minor, then further investigation is needed before anything can be said.

Note that a bordered matrix has a total of \( n+m \) leading principal minors. Thus if we compute the last \( n-m \), we are omitting the first \( 2m \) leading principal minors, and starting at the \((2m+1)\)th leading principal minor. As before, for positive definiteness, we require that all leading principal minors starting with the \((2m+1)\)th have the sign of \((-1)^m\). For negative definiteness, we require that the \((2m+1)\)th leading principal minor have the sign of \((-1)^{m+1}\) and that all larger leading principal minors alternate in sign.

Exercise. Derive this sign restriction on leading principal minors for the negative definite case from the earlier statement.

3. (Quasi-)Concavity and (Quasi-)Convexity

3a) Consider the function \( f:C \rightarrow \mathbb{R} \), where \( C \) is an open convex subset of \( \mathbb{R}^n \). If \( f \) has continuous second order partial derivatives, we may define a **Hessian** matrix as follows:

\[
H \equiv \begin{bmatrix}
f_{11} & f_{12} & \cdots & f_{1n} \\
f_{21} & f_{22} & \cdots & f_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix}
\]

where

\[f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}\]

Then \( f \) is concave if and only if the Hessian is negative semi-definite for all \( x \in C \) and \( f \) is convex if and only if the Hessian is positive semi-definite for all \( x \in C \). A sufficient (but not necessary) condition for \( f \) to be strictly concave is that the Hessian be negative definite for all \( x \in C \). A sufficient (but not necessary) condition for \( f \) to be strictly convex is that the Hessian be positive definite for all \( x \in C \).

3b) Let \( f \) be as defined in 3a). If \( f \) has continuous second order partial derivatives, we may define the following bordered matrix:

\[
K = \begin{bmatrix}
0 & f_1 & \cdots & f_n \\
- & - & \cdots & - \\
f_1 & f_{11} & \cdots & f_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
f_n & f_{n1} & \cdots & f_{nn}
\end{bmatrix}
\]
A sufficient (but not necessary) condition for $f$ to be strictly quasi-concave is that $K$ satisfy the conditions for negative definiteness of a bordered matrix for all $x \in C$, i.e., the 3rd order leading principal minor must be positive and all larger leading principal minors must alternate in sign.

A sufficient (but not necessary) condition for $f$ to be strictly quasi-convex is that $K$ satisfy the conditions for positive definiteness of a bordered matrix for all $x \in C$, i.e., the 3rd order and all larger leading principal minors must be negative.

If the strict inequalities are replaced by weak inequalities, then one gets a necessary condition for quasi-concavity/convexity.

3c) Given a convex set $C$, its closure $\text{cl } C$ consists of the union of $C$ and its boundary points.\(^1\) We have the following result: If $f$ is continuous on $\text{cl } C$ and concave/convex/quasi-concave/quasi-convex on $C$, then it is likewise concave/convex/quasi-concave/quasi-convex (respectively) on $\text{cl } C$. The value of this result is that it means that results obtained for $\mathbb{R}^{n+}_{++}$ can be extended to $\mathbb{R}^n_{++}$.

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Quadratic Forms Questions

For the current topic, the questions in Simon and Blume are similar to those in my own “question bank” so I have only set Simon and Blume questions, with none of my own.

From Simon and Blume, do the following:

- Chapter 16: 16.1, 16.6.

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\(^1\) Or, equivalently, the union of $C$ and its accumulation points or, equivalently, the intersection of all closed sets containing $C$. 