1a) Given two sets $X$ and $Y$, a function from $X$ to $Y$, written $f: X \to Y$, is a rule that associates with each element $x \in X$ a unique element $f(x) \in Y$. The set $X$ is called the domain of the function. The set $Y$ goes under various names including codomain, target and target space. The element $f(x)$ is called the value of the function at $x$ or the image of $x$ under $f$. The set of all the $f(x)$ values is called the image or the range of the function, denoted $f(X)$, i.e., $f(X) = \{ f(x) \in Y \mid x \in X \}$. The range may be a proper subset of the codomain, e.g., if $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2$, the range of the function is $\mathbb{R}_+$ which is a proper subset of the codomain $\mathbb{R}$. If the range and codomain are equal, then the function is said to map $X$ onto $Y$ (the general statement is that the function maps $X$ into $Y$ — onto is a special case of into). If $f$ maps $X$ onto $Y$, then $f(x) = y$ has a solution for $x$ for each $y \in Y$.

1b) In working with functions, it is important to be clear on the notational conventions that govern them. Suppose, for example, that we have a function $f: X \to Y$. In the expression $f(x)$, the letter $x$ is simply any element belonging to the domain $X$ and $f(x)$ is the element of the codomain $Y$ that results from applying the function rule $f$ to that element of the domain. The fact that the letter $x$ is used has no significance. Suppose, for example, that we have a function $f$ with a domain given by $X = \{ x \in \mathbb{R} \mid 0 \leq x \leq 10 \}$ and that $f(x) = x^2$ for all $x$ in the domain. Then it is equally true that $f(y) = y^2$ for all $y \in X$, $f(z) = z^2$ for all $z \in X$ and $f(a - 1) = (a - 1)^2$ for all $(a - 1) \in X$. Note that in this last example $(a - 1) \in X$ iff $1 \leq a \leq 11$. More generally, given a function $f: X \to Y$, it is permissible to replace the $x$ in $f(x)$ by any object $\triangle$ whatsoever provided $\triangle \in X$. Assuming this condition is met, $f(\triangle)$ is given by the following condition: if $\triangle = x$, then $f(\triangle) = f(x)$. If $f(x)$ is given by a formula, then this means that $f(\triangle)$ is calculated by replacing $x$ by $\triangle$ wherever $x$ appears in the formula for $f(x)$, e.g., if

$$f(x) = \frac{x^2 - 3x + 9}{8 - 2x^2}$$

for all $x \neq 2, -2$, then

$$f(\triangle) = \frac{\triangle^2 - 3\triangle + 9}{8 - 2\triangle^2}$$

for all $\triangle \neq 2, -2$.

For example, if $\triangle = (a^2 - 2b + c)$, then

$$f(a^2 - 2b + c) = \frac{[a^2 - 2b + c]^2 - 3[a^2 - 2b + c] + 9}{8 - 2[a^2 - 2b + c]^2}$$

for all $a^2 - 2b + c \neq 2, -2$. 
1c) A function of \( n \) variables can be represented by \( f: X \to Y \) where \( X \) is a subset of \( \mathbb{R}^n \) and \( Y \) is a subset of \( \mathbb{R} \). Economic examples include the production function \( Q = F(K, L) \) (where \( K \) is the quantity of capital and \( L \) is the quantity of labour) and the utility function \( U = U(x_1, x_2, \ldots, x_n) \) (where \( x_1, x_2, \ldots, x_n \) are quantities of goods \( 1, \ldots, n \)). Using vector notation, the utility function may be written more succinctly as \( U = U(\mathbf{x}) \), where \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \).

1d) Just as with functions of a single variable, it is important to understand the notation for functions of several variables. Suppose, for example, that we have a function \( f \) with a domain given by \( X = \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0 \} \). Then the domain is to be interpreted as consisting of all 3-dimensional vectors with non-negative components, not just all \( x \) vectors with non-negative components. Thus if \( \mathbf{y} = [7, 0, 4] \), then that \( \mathbf{y} \) vector belongs to the function’s domain.

Similarly, in the expression \( f(\mathbf{x}) \), the letter \( \mathbf{x} \) is simply any element belonging to the domain \( X \), and \( f(\mathbf{x}) \) is the element of the codomain \( Y \) that results from applying the function rule \( f \) to that element of the domain. For any \( \triangle \in X \), we may calculate \( f(\triangle) \) by applying the rule: if \( \triangle = \mathbf{x} \), then \( f(\triangle) = f(\mathbf{x}) \). If \( f(\mathbf{x}) \) is given by a formula, then this means that \( f(\triangle) \) is calculated by replacing the \( i \)th component of \( \mathbf{x} \) (i.e., \( x_i \)) by the \( i \)th component of \( \triangle \) wherever \( x_i \) appears in the formula for \( f(\mathbf{x}) \), e.g., if

\[
f(x_1, x_2, x_3) = x_1^2 - 5x_2 + 9x_3 + e^{x_1}
\]

for all \((x_1, x_2, x_3) \in \mathbb{R}^3\)

then

\[
f(7, a^2, y) = 7^2 - 5a^2 + 9y + e^7
\]

for all \((7, a^2, y) \in \mathbb{R}^3\).

1e) As a further illustration of the generality of the function concept, consider the following. Suppose that we have two functions, each of three variables:

\[
y^1 = f^1(x_1, x_2, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{1}
\]

\[
y^2 = f^2(x_1, x_2, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{2}
\]

Then, as an alternative to considering these as two functions, we can think of them as just one function \( f: \mathbb{R}^3 \to \mathbb{R}^2 \) which associates with each element \((x_1, x_2, x_3)\) in the domain of \( \mathbb{R}^3 \) a unique element \((y^1, y^2)\) in the codomain of \( \mathbb{R}^2 \). Such functions are sometimes termed “vector-valued”. Using vector notation, we may write (1) and (2) as \( y = f(\mathbf{x}) \), where \( f \) is a vector-valued function.

1f) In economics we frequently encounter composite functions or “functions of functions”. These are formally defined as follows:

Given \( f: X \to Y \) and \( g: Y^* \to Z \), where \( Y^* \) includes \( f(X) \) (i.e., the domain of \( g \) includes the range of \( f \)), we may define the composite function, \( h: X \to Z \), where

\[
h(x) \equiv g(f(x)) \quad \text{for all } x \in X.
\]
This is illustrated in Figure 1 (the domain of \( g \) has not been shown explicitly, but is to be understood as including all \( f(x) \) values):

![Diagram](image)

**Figure 1**

e.g., if \( X = Y = Y^* = Z = \mathbb{R} \), \( f(x) = 3x \) and \( g(y) = 2 + y^2 \), then we may define the composite function, \( h: X \rightarrow Z \), where

\[
h(x) = g(f(x)) \\
= g(3x) \\
= 2 + (3x)^2.
\]

Note that, rather than introduce a new letter such as \( h \) to denote the composite function, an alternative is to use \( g \circ f \), which makes clearer how the values of the composite function are calculated. Thus, instead of \( h: X \rightarrow Z \), we can write \( g \circ f: X \rightarrow Z \).

As an economic example, suppose that utility is a function of the quantity consumed of each of \( n \) consumption goods, i.e., we have

\[
U = g(x_1, \ldots, x_n).
\]

Further suppose that the demand for good \( i \) depends on the prices of goods 1 to \( n \) and on money income \( M \), i.e.,

\[
\begin{align*}
x_1 &= f^1(p_1, \ldots, p_n, M) \\
x_2 &= f^2(p_1, \ldots, p_n, M) \\
&\vdots \\
x_n &= f^n(p_1, \ldots, p_n, M).
\end{align*}
\]

These demand functions can be regarded as a single vector-valued function \( f \) with a domain of \( \mathbb{R}_{+}^{n+1} \), a codomain of \( \mathbb{R}^n \) and values given by \( \mathbf{x} = f(\mathbf{p}, M) \).

Utility is determined directly by the quantities of goods consumed but, since these consumption quantities are determined by prices and income, it follows that utility is determined *indirectly* by prices and income. We represent this mathematically by the use of a composite function. In effect this involves substituting the demand functions into the utility function.
Formally, we have a demand function \( f: \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n \) and a utility function \( g: \mathbb{R}_+^n \to \mathbb{R} \). From these two we form the composite function \( h: \mathbb{R}_+^{n+1} \to \mathbb{R} \), where

\[
h(p_1, \ldots, p_n, M) = g\left( f_1(p_1, \ldots, p_n, M), \ldots, f_n(p_1, \ldots, p_n, M) \right)
\]

or, more succinctly,

\[
h(p, M) = g(f(p, M)).
\]

This gives utility as a function of prices and income. The composite function \( h(p, M) \) is known as the “indirect utility function”.

1g) Note that a composite function may involve more than two functions. One could have, say, four functions, \( a, b, c \) and \( d \) and form the composite function: \( h(x) = a(b(c(d(x)))) \).

1h) Let \( f \) and \( g \) be two scalar-valued functions. Then the sum, difference, product, quotient, minimum and maximum of \( f \) and \( g \) are defined as follows:

- Sum: \( (f + g)(x) = f(x) + g(x) \)
- Difference: \( (f - g)(x) = f(x) - g(x) \)
- Product: \( (f \cdot g)(x) = f(x) \cdot g(x) \)
- Quotient: \( (f/g)(x) = f(x)/g(x) \)
- Minimum: \( \min\{f, g\}(x) = \min\{f(x), g(x)\} \)
- Maximum: \( \max\{f, g\}(x) = \max\{f(x), g(x)\} \).

The domain in all cases is the intersection of the domains of \( f \) and \( g \), subject to the qualification that the quotient function is only defined for those values of \( x \) for which \( g(x) \neq 0 \).

1i) The function \( f \) associates with each \( x \in X \) a unique \( y \in Y \), but it need not do the reverse, i.e., associated with a \( y \in Y \) may be more than one value of \( x \in X \), e.g., if \( f(x) = x^2 \), then each positive \( y \) is associated with two \( x \) values, \( \pm \sqrt{y} \). The inverse image of \( y \in Y \) is defined as the set of all \( x \in X \) such that \( f(x) = y \):

\[
f^{-1}(y) = \{ x \mid x \in X, f(x) = y \}
\]

In the example described above,

\[
f^{-1}(y) = \{ x \mid x \in \mathbb{R}, x^2 = y \}
\]

Note that the inverse image may be an empty set. In the set just defined, \( f^{-1}(y) = \{ \pm \sqrt{y} \} \) for \( y \geq 0 \), but \( f^{-1}(y) = \emptyset \) for \( y < 0 \).

The inverse image of a subset of the codomain, \( Y' \subset Y \), can also be defined as follows:

\[
f^{-1}(Y') = \{ x \mid x \in X, f(x) \in Y' \},
\]

e.g., if \( Y' = \{ y \mid 1 < y < 4 \} \) and \( f(x) = x^2 \), then

\[
f^{-1}(Y') = \{ x \mid 1 < x < 2 \text{ or } -2 < x < -1 \}.
\]
1j) If for each \( y \in Y \), the inverse image consists of at most one point (recall that it may be an empty set), then the function is said to be one to one. Another way of saying this is that, if \( f \) is one to one, then the equation \( f(x) = y \) (where \( y \in Y \)) has at most one solution for \( x \). Yet another way of defining a one to one function is to say that \( f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \) for all \( x_1, x_2 \in X \).

1k) If \( f \) is both one to one and onto, then there exists an inverse function \( f^{-1}: Y \to X \), i.e., \( f^{-1} \) associates with each \( y \in Y \) a unique value \( f^{-1}(y) = x \in X \), where \( f^{-1}(y) \) is the inverse image defined earlier, i.e.,

\[
 f^{-1}(y) = \{ x \mid x \in X, f(x) = y \}
\]

1l) If \( y \) is in the range of \( f \), then

\[
 f \left( f^{-1}(y) \right) = y.
\]

However, it is not necessarily true that for \( x \in X \):

\[
 f^{-1} \left( f(x) \right) = x.
\]

It is true if \( f \) is one to one. In the general case, we have

\[
 f^{-1} \left( f(x) \right) \ni x,
\]

but \( f^{-1} \left( f(x) \right) \) may include other elements in addition to \( x \).

2. Limits and Continuity

2a) Consider a function \( f: X \to Y \), where \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \). Let \( x_0 \in \mathbb{R}^n \) and \( s \in \mathbb{R}^m \). Note that it is possible but not required that \( x_0 \in X \) and \( s \in Y \). We do assume, however, that \( x_0 \) is an accumulation point of \( X \).

If the limit of \( f \) as \( x \) approaches \( x_0 \) equals \( s \), then we denote this with the following notational forms:

\[
 \lim_{x \to x_0} f(x) = s \quad \text{or} \quad f(x) \to s \text{ as } x \to x_0.
\]

This limit relation holds iff the following condition is met: For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
 \| f(x) - s \| < \epsilon \quad \text{for all } x \in X \text{ satisfying } 0 < \| x - x_0 \| < \delta.
\]

For example, if \( X = Y = \mathbb{R} \) and \( f(x) = 2x \), then

\[
 \lim_{x \to 3} f(x) = 6
\]

1 The point \( x_0 \in \mathbb{R}^n \) is an accumulation point of \( X \) if every \( \epsilon \)-ball about \( x_0 \) contains at least one point \( x \in X \), where \( x \neq x_0 \).
since, if we set $\delta = \epsilon/2$, then

$$\|2x - 6\| < \epsilon \quad \text{for all } x \in \mathbb{R} \text{ satisfying } 0 < \|x - 3\| < \delta = \epsilon/2,$$

so condition (3) is satisfied.

2b) Note that the condition (3) specifies that $\|x - x_0\| > 0$. Thus the value of $f(x)$ at $x = x_0$ plays no role in the definition of the limit — as noted earlier, it is not even necessary that $x_0$ be in the domain of $f$ or that $s$ be in its codomain.

2c) The limit of a function may or may not exist. It may fail to exist if the function “jumps” at $x_0$ or if it oscillates “infinitely rapidly” at $x_0$.

2d) An alternative but equivalent definition of the limit of a function may be given in terms of sequences. Let $f$, $x_0$, and $s$ be as defined above. Then

$$\lim_{x \to x_0} f(x) = s$$

iff, for every sequence $\{x_k\}$ in $X$ satisfying $x_k \neq x_0$ (all $k$) and $x_k \to x_0$, the sequence $\{f(x_k)\}$ has a limit of $s$, i.e.,

$$\lim_{k \to \infty} f(x_k) = s \quad \text{or} \quad f(x_k) \to s \text{ as } k \to \infty.$$

Note that, as in the earlier definition, the value of $f$ at $x_0$ is irrelevant.

2e) As with limits, we can give two definitions of a continuous function. Define $f$ as above and assume $x_0 \in X$. Then $f$ is said to be continuous at $x_0$ iff, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x) - f(x_0)\| < \epsilon \quad \text{for all } x \in X \text{ satisfying } \|x - x_0\| < \delta. \quad (4)$$

Plainly, unlike in the definition of the limit of a function given above, it is necessary for continuity at $x_0$ that $f(x)$ be defined at $x = x_0$ (to be “defined” in this context, $f(x_0)$ must be finite) and the value of $f(x_0)$ is fundamental to the definition.

2f) The sequence definition of continuity is as follows: Define $f$ as above and assume $x_0 \in X$. Then $f$ is said to be continuous at $x_0$ iff, for every sequence $\{x_k\}$ in $X$ with a limit of $x_0$, the sequence $\{f(x_k)\}$ has a limit of $f(x_0)$, i.e.,

$$\lim_{k \to \infty} f(x_k) = f(x_0) \quad \text{or} \quad f(x_k) \to f(x_0) \text{ as } k \to \infty.$$

Note that, unlike in the limit of a function definition, we do not require that $x_k \neq x_0$, i.e., we admit a wider class of sequences.

2g) As the above definitions will suggest, the continuity of a function is closely related to the concept of a function’s limit. We may formalise this as follows. Defining $f$ as above, if $x_0$ is an accumulation point of $X$, then $f$ is said to be continuous at $x_0$ iff

$$\lim_{x \to x_0} f(x) = f(x_0).$$
If $x_0 \in X$ but $x_0$ is not an accumulation point (i.e., it is an isolated point), then $f$ is automatically continuous at $x_0$.

Note that this relationship can be used in two ways: if we know the limit, then we can test for continuity; on the other hand, if we know that a function is continuous, then the relationship provides an easy way to find the limit.

2h) The function $f: X \rightarrow Y$ is said to be continuous iff it is continuous at every point $x_0 \in X$.

2i) As before, consider a function $f: X \rightarrow Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Let $Y_i$ be the set of possible values of the $i$th component of a vector belonging to $Y$, i.e.,

$$Y_i = \{ y \in \mathbb{R} \mid y \in Y, y = y_i \}.$$  

Then $f$ is continuous iff its component functions $f_i: X \rightarrow Y_i$ are continuous for $i = 1 \ldots m$.

2j) Let $f$ and $g$ be two functions from $X$ to $Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and assume:

$$\lim_{x \to x_0} f(x) = a \quad \text{and} \quad \lim_{x \to x_0} g(x) = b.$$

Then

$$\lim_{x \to x_0} (f + g)(x) = a + b$$
$$\lim_{x \to x_0} (f - g)(x) = a - b$$
$$\lim_{x \to x_0} (f \cdot g)(x) = a \cdot b.$$

Further, if $m = 1$, then

$$\lim_{x \to x_0} (f/g)(x) = a/b \quad \text{for } b \neq 0$$
$$\lim_{x \to x_0} \min\{f, g\}(x) = \min\{a, b\}$$
$$\lim_{x \to x_0} \max\{f, g\}(x) = \max\{a, b\}.$$

2k) Let $f$ and $g$ be as defined above. If $f$ and $g$ are both continuous, then the functions formed by taking the sum, difference, inner product, and (for $m = 1$) quotient, minimum and maximum of these functions are also continuous on the domain for which the new function is defined (in the quotient case, this domain will exclude any points at which the function in the denominator is zero).

2l) Given $f: X \rightarrow Y$ and $g: V \rightarrow Z$, where $f(X) \subset V$, we may define the composite function, $h: X \rightarrow Z$, where

$$h(x) \equiv g(f(x)) \quad \text{for all } x \in X,$$

e.g., if $f(x) = 2x$ and $g(y) = y^2$, then

$$h(x) = g(f(x)) = (2x)^2.$$

If $f$ and $g$ are both continuous functions, then the composite function $h$ is also continuous.
3. Continuity and Sets

3a) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Then \( f \) is continuous iff, for every open set \( V \subset \mathbb{R}^m \), the inverse image \( f^{-1}(V) \) is an open set. Similarly, \( f \) is continuous iff, for every closed set \( Y \subset \mathbb{R}^m \), the inverse image \( f^{-1}(Y) \) is a closed set.

3b) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). If \( f \) is continuous and \( X \subset \mathbb{R}^n \) is compact, then \( f(X) \) is compact.

3c) Intermediate Value Theorem. Let \( f \) be a real scalar-valued function which is continuous at each point on the closed interval \([a, b]\). Choose two arbitrary points \( x_1 < x_2 \) in \([a, b]\) such that \( f(x_1) \neq f(x_2) \). Then \( f \) takes on every value between \( f(x_1) \) and \( f(x_2) \) somewhere on the interval \((x_1, x_2)\).

3d) Weierstrass’s Theorem. Let \( S \) be a compact subset of \( \mathbb{R}^n \) and let \( f \) be a real scalar-valued function which is continuous at each point on \( S \). Then \( f \) has both a minimum and a maximum value on \( S \), i.e., there exist \( x^a \) and \( x^b \) belonging to \( S \) such that \( f(x^a) \leq f(x) \leq f(x^b) \) for all \( x \in S \).

4. Homogeneous and Homothetic Functions

4a) A set \( X \subset \mathbb{R}^n \) is a cone iff \( tx \in X \) implies that \( tx \in X \) for all \( t \in \mathbb{R}_{++} \).

4b) A function \( f : X \to I \) (where \( X \subset \mathbb{R}^n \), \( X \) is a cone and \( I \subset \mathbb{R} \)) is homogeneous of degree \( k \) iff
\[
f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n) \quad \text{for all } x \in X \text{ and } t \in \mathbb{R}_{++}.
\]

4c) Let \( I, J \subset \mathbb{R} \). Then a function \( g : I \to J \) is a monotonic transformation iff \( g \) is a strictly increasing function, i.e., iff \( x > y \) implies \( g(x) > g(y) \).

4d) A function is homothetic iff it is monotonic transformation of a homogeneous function. Formally, assume \( X \subset \mathbb{R}^n \), \( X \) is a cone and \( I, J, K \subset \mathbb{R} \). Then \( h : X \to J \) is homothetic iff there exists a homogeneous function \( f : X \to I \) and a monotonic transformation \( g : K \to J \) (where \( f(I) \subset K \)) such that \( h(x) = g \circ f(x) \) for all \( x \in X \).

This is of interest because a monotonic transformation of a function has the same level curves (indifference curves, isoquants or whatever) as the original function.

5. Convex Sets

5a) A set \( C \subset \mathbb{R}^n \) is convex iff for every \( x \in C \) and \( y \in C \), the vector \( z = \lambda x + (1 - \lambda)y \) also belongs to \( C \) for every \( \lambda \) satisfying \( 0 < \lambda < 1 \).

Geometrically, this means that, for every pair of elements of \( C \), a straight line between the points lies wholly within \( C \).
An expression of the form $\lambda x + (1 - \lambda)y$, where $0 \leq \lambda \leq 1$, is known as a convex combination of $x$ and $y$.

**Note.** A set which is not convex is simply called non-convex. The opposite of a convex set is not a concave set: there is no such thing as a concave set.

5b) By convention, the empty set is convex.

5c) The intersection of a finite or infinite number of convex sets is also convex.

5d) The sum of a finite number of convex sets is a convex set.

5e) Any scalar multiple of a convex set is a convex set.

5f) The cartesian product of a finite number of convex sets is a convex set.

5g) The union of two (or more) convex sets is not necessarily convex.

5h) A convex set $C$ is a convex cone iff, in addition to satisfying the requirements for a convex set, it has the property that $x \in C$ implies that $tx \in C$ for all $t > 0$.

A set $C$ is strictly convex iff, for every $x \in C$ and $y \in C$, where $x \neq y$, the element $z = \lambda x + (1 - \lambda)y$ belongs to interior of $C$ for all $\lambda$ satisfying $0 < \lambda < 1$.

6. Concave/Convex and Quasi-Concave/Quasi-Convex Functions

6a) Definitions of the four function types are given below. These definitions all depend upon the domains of the functions being convex sets (if the domains are not convex, then the definitions cannot be applied). Accordingly, it is to be understood in all of the definitions to follow that the functions’ domains are convex sets.

1. A function $y = f(x)$ with a domain $X$ is concave if and only if for every $x^a, x^b \in X$:

$$f(\lambda x^a + (1 - \lambda)x^b) \geq \lambda f(x^a) + (1 - \lambda)f(x^b) \quad \text{for all } 0 < \lambda < 1.$$

2. A function $y = f(x)$ with a domain $X$ is convex if and only if for every $x^a, x^b \in X$:

$$f(\lambda x^a + (1 - \lambda)x^b) \leq \lambda f(x^a) + (1 - \lambda)f(x^b) \quad \text{for all } 0 < \lambda < 1.$$

3. A function $y = f(x)$ with a domain $X$ is quasi-concave if and only if for every $x^a, x^b \in X$:

$$f(\lambda x^a + (1 - \lambda)x^b) \geq \min \left\{ f(x^a), f(x^b) \right\} \quad \text{for all } 0 < \lambda < 1.$$

4. A function $y = f(x)$ with a domain $X$ is quasi-convex if and only if for every $x^a, x^b \in X$:

$$f(\lambda x^a + (1 - \lambda)x^b) \leq \max \left\{ f(x^a), f(x^b) \right\} \quad \text{for all } 0 < \lambda < 1.$$

If the weak inequality in the above definitions is replaced by a strict inequality and if we add the condition that $x^a \neq x^b$, then we get the definition of strict concavity, strict convexity, strict quasi-concavity, and strict quasi-convexity respectively.
Figure 2

Figure 3
Quasi-Concave Function

\[ f(\lambda x^a + (1-\lambda)x^b) \]

\[ f(x^a) \]

\[ \min\{f(x^a), f(x^b)\} = f(x^a) \]

Figure 4

Quasi-Convex Function

\[ f(\lambda x^a + (1-\lambda)x^b) \]

\[ f(x^a) \]

\[ \max\{f(x^a), f(x^b)\} = f(x^b) \]

Figure 5
6b) Every concave (convex) function is quasi-concave (quasi-convex), but the converse is not true, i.e., concavity/convexity is a stronger (i.e., more restrictive) assumption than quasi-concavity/quasi-convexity, e.g.,

\[ f(x) = \begin{cases} x^2 & \text{for all } x \leq 0 \\ 0 & \text{for all } x > 0 \end{cases} \]

is quasi-concave but not concave.

6c) If \( f \) is a (strictly) concave function, then \( -f \) is a (strictly) convex function and \textit{vice versa}. Similarly, if \( f \) is a (strictly) quasi-concave function, then \( -f \) is a (strictly) quasi-convex function and \textit{vice versa}.

6d) A linear function is both concave and convex.

6e) A monotonic transformation of a quasi-concave(convex) function is itself quasi-concave(convex). By contrast, a monotonic transformation of a concave(convex) function need not be concave(convex).

6f) Any non-negative linear combination of concave (convex) functions is concave (convex), i.e., if

\[ F(x) = \sum_{i=1}^{n} \lambda_i f^i(x) \quad \text{where } \lambda_i \geq 0 \text{ for all } i, \]

then \( F(x) \) is concave (convex) if \( f^i(x) \) is concave (convex) for all \( i \).

By contrast, a non-negative linear combination of quasi-concave (quasi-convex) functions is \textit{not} necessarily quasi-concave (quasi-convex), e.g. the two functions

\[ f^1(x) = \begin{cases} x^2 & \text{for all } x \leq 0 \\ 0 & \text{for all } x > 0 \end{cases} \quad \text{and} \quad f^2(x) = \begin{cases} 0 & \text{for all } x \leq 0 \\ x^2 & \text{for all } x > 0 \end{cases} \]

are both quasi-concave, but their sum

\[ F(x) = f^1(x) + f^2(x) = x^2 \]

is not quasi-concave.

6g) If \( f \) is a concave/convex function, then it is continuous on the interior of its domain.

7. The Relationship Between Function Type and Convex Sets

Another way of characterising a function’s type — i.e., whether it is concave, convex, quasi-concave or quasi-convex — involves the use of convex sets. This can be a little confusing since people tend to feel that convex functions should be related to convex sets and concave functions should be related to concave sets. However, since there is no such thing as a concave set, this is obviously impossible. Instead, \textit{both} convex and concave functions (and the corresponding “quasi-” functions) are defined in terms of convex sets.
Specifically, a function $f(x)$ is convex/concave/quasi-convex/quasi-concave if and only if a particular set defined in terms of $f(x)$ is convex. Four differently defined sets are used in this test — one for each function type. These set definitions are given below. It is to be understood in each case that the function has a convex domain, denoted by $X$.

1. A function $f(x)$ is concave if and only if the set
   \[ T_1 = \{ (x, y) \mid x \in X, y \leq f(x) \} \]
   is convex.

2. A function $f(x)$ is convex if and only if the set
   \[ T_2 = \{ (x, y) \mid x \in X, y \geq f(x) \} \]
   is convex.

3. A function $f(x)$ is quasi-concave if and only if the set
   \[ T_3 = \{ x \mid x \in X, f(x) \geq k \} \]
   is convex for each real number $k$.

4. A function $f(x)$ is quasi-convex if and only if the set
   \[ T_4 = \{ x \mid x \in X, f(x) \leq k \} \]
   is convex for each real number $k$.

**Note.** While it is possible to relate strict concavity etc. to strictly convex sets, it is rather tricky and is therefore omitted here.