1. See Lecture 6.

2. (a) \( f \) is homothetic, since

\[
\exp\left( -f(x, y) \right) = x^2 + y^2
\]

is homogeneous, but \( f \) itself is not homogenous.

(b) \( g(x, y) = e^{\ln(x^2) - 2\ln(y)} = x^2/y^2 \) is homogeneous of degree 0, hence also homothetic.

(c) \( h \) is neither homogenous nor homothetic.

3. (a) The consumer’s utility

\[
u(x, y) = u\left(x, \frac{M - p_x x}{p_y}\right) = x^{1/4} \left( \frac{M - p_x x}{p_y} \right)^{3/4}\]

is maximized at a point where

\[
0 = \frac{d}{dx} \left[ u\left(x, \frac{M - p_x x}{p_y}\right) \right] = \frac{1}{4} x^{-3/4} \left( \frac{M - p_x x}{p_y} \right)^{3/4} + x^{1/4} \cdot \frac{3}{4} \left( \frac{M - p_x x}{p_y} \right)^{-1/4} \left( -\frac{p_x x}{p_y} \right)
\]

\[
= \frac{1}{4} \left( \frac{M - p_x x}{p_y} \right)^{3/4} - \frac{3}{4} \frac{p_x}{p_y} \left( \frac{M - p_x x}{p_y x} \right)^{-1/4}
\]

\[
\Rightarrow 3p_x = \frac{M - p_x x}{p_y x} \quad \Rightarrow \quad x = \frac{M}{4p_x}
\]
\[ \epsilon_{x,p_x} = \frac{p_x}{x} \cdot \frac{dx}{dp_x} = \frac{p_x}{(M/4p_x)} \cdot \left( -\frac{M}{4} \right) p_x^{-2} = -1 \]

\[ \epsilon_{x,p_y} = \frac{p_y}{x} \cdot \frac{dx}{dp_y} = \frac{p_x}{x} \cdot 0 = 0. \]

4. (a) Revenue = \( pq = (10 - q)q \)

\[ MR(q) = \frac{d}{dq}[\text{revenue}] = 10 - 2q \]

(b) \[ q = 1 - \frac{1}{1 + L} \Rightarrow \frac{1}{1 + L} = 1 - q \Rightarrow L = \frac{1}{1 + q} - 1 = \frac{q}{1 - q} \]

\[ C(q) = wL = \frac{wj}{1 - q} \]

(c) \[ MC(q, w) = \frac{w}{1 - q} - \frac{(-1)wq}{(1-q)^2} = w \frac{1}{(1-q)^2} \]

At a profit maximizing quantity we must have

\[ 10 - 2q = MR(q) = MC(q, w) = \frac{w}{(1-q)^2} \]

(d) \[ f(w, g) = MR(q) - MC(q, w) = 10 - 2q - \frac{w}{(1-q)^2} \]

Let \( g(w) \) be the profit maximizing productivity. Thus

\[ \frac{dg}{dw} = -\frac{2f}{2g} = -\frac{1}{(1-q)^2} + \frac{w}{(1-q)^3} (-1) = \frac{1 - q}{2[(1-q)^3 + w]} \]

[We could solve for \( w \) as a function of \( q \), then substitute.]

5. First observe that \( C_3(S) \) is convex, since

\[ (1 - \gamma)[\alpha_1 x^1 + \ldots + \alpha_h x^h] + \gamma[\beta_1 y^1 + \ldots + \beta_\ell y^\ell] \]

\[ = (1 - \gamma)\alpha_1 x^1 + \ldots + (1 - \gamma)\alpha_h x^h + \gamma\beta_1 y^1 + \ldots + \gamma\beta_\ell y^\ell \]
is an element of $C_3(S)$, in that $x^1, \ldots, x^h, y^1, \ldots, y^\ell \in S$ and

$$(1 - \gamma)\alpha_1 + \ldots + (1 - \gamma)\alpha_k + \gamma\beta_1 + \ldots + \gamma\beta_\ell$$

$$= (1 - \gamma)\left(\sum_{j=1}^k \alpha_j\right) + \gamma\left(\sum_{j=1}^\ell \beta_j\right)$$

$$= (1 - \gamma) + \gamma = 1.$$

Since $S \subset C_3(S)$, the definition of $C_1(S)$ now implies that $C_1(S) \subset C_3(S)$.

Second, we will show that any convex set containing $S$ must contain $C_3(S)$. This follows by induction on $k$, in that if any convex set containing $S$ contains all combinations of the form $\alpha_1x^1 + \ldots + \alpha_{k-1}x^{k-1}$ as per the definition of $C_3(S)$, then, by the definition of convexity, it must also contain all combinations of the form

$$\alpha_1x^1 + \ldots + \alpha_kx^k =$$

$$(1 - \alpha_k)\left(\frac{\alpha_1}{1 - \alpha_k}x^1 + \ldots + \frac{\alpha_{k-1}}{1 - \alpha_k}x^{k-1}\right) + \alpha_kx_k.$$

Thus $C_3(S) \subset C_2(S)$.

Finally we observe that $C_1(S)$ is one of the sets whose intersection is $C_2(S)$, so that we have $C_2(S) \subset C_1(S)$.

We have shown that $C_1(S) \subset C_3(S) \subset C_2(S) \subset C_1(S)$, so all three sets are equal.