I. Introduction

A. The problem studied here is to devise a tax system that raises a certain amount of revenue in a way that minimizes the loss to the economy due to distorted incentives.

B. On the one hand this is a nice application of the Kuhn-Tucker theorem.

C. The model also has important implications for policy.

II. The Economy

A. To begin with we consider an economy consisting of a single agent. (Really, we are thinking of a large number of identical agents.)

1. In this way we avoid issues having to do with the possibility that taxes might redistribute income from one group to another.

B. The individual’s initial endowment consists of $e$ units of labor, which is the single input to production processes.

C. There are $L$ other goods which are produced from labor using constant returns to scale technologies.

1. For $\ell = 1, \ldots, L$ let $c_\ell$ be the amount of labor required to produce one unit of good $\ell$.

D. The consumer’s utility function is

$$u(z, x) = z + \sum_{\ell=1}^{L} \phi_\ell(x_\ell)$$
where, for each $\ell$, $\phi_\ell : \mathbb{R}^+ \to \mathbb{R}$ is increasing, $C^1$, and differentially strictly concave, meaning that the second derivative of $\phi_\ell$ is everywhere negative.

2. We also assume that, for each $\ell$, $\lim_{x_\ell \to \infty} \frac{d\phi_\ell}{dx_\ell}(x_\ell) = 0$.

E. Since the utility function is additive across goods, for each good the consumer’s inclination is to buy up to the point where the marginal utility is equal to the price. Define the function $f_\ell : \mathbb{R}_{++} \to \mathbb{R}$ implicitly by the equation

$$\frac{d\phi_\ell}{dx_\ell}(f_\ell(p_\ell)) = p_\ell.$$

1. It simplifies the analysis to assume that

$$\sum_{\ell=1}^{L} f_\ell(c_\ell) < c,$$

which means that the consumer will not “run out of hours in the day” even if all goods are priced at cost.

a. Typically one imagines that the goods that one is considering imposing special taxes on (e.g., gasoline, alcohol, air travel) do not constitute a large fraction of total expenditure.

2. We also assume that $f_\ell(c_\ell) > 0$ for all $\ell$, meaning that there are no goods for which there is no demand when they are priced at cost.

III. The Government

A. We will assume, without modelling it, that the government needs to raise an amount of revenue sufficient to purchase $R$ units of labor.

B. We will also assume that the only available methods of raising this revenue are ad valorem taxes on the transactions in the economy, which are the sale of the agent’s labor to the firms and government, and the sales of produced goods to the agent.

1. “Ad valorem” means in proportion to value. That is, the tax rate has to be the same independent of the amount transacted.
2. This rules out things like the progressive income tax.

C. From the point of view of the consumer, the effective tax rate on any good is the one that results from the combination of the tax on labor income and the tax specific to that good.

1. This means that a tax on labor income is equivalent to a uniform tax on all produced goods.

2. Since any tax system is equivalent to one in which only produced goods are taxed, we can and shall restrict attention to systems in which the tax on labor is zero.

IV. The Maximization Problem and Its Solution

A. The constraints are

\[ 0 \geq -e + z + \sum_{\ell=1}^{L} c_\ell x_\ell; \]

\[ 0 \geq x_\ell - f_\ell(c_\ell + t_\ell) \text{ for } \ell = 1, \ldots, L; \]

\[ 0 \geq R - \sum_{\ell=1}^{L} t_\ell x_\ell. \]

B. The Lagrangean is

\[ \mathcal{L}(z, x, t) = z + \sum_{\ell=1}^{L} \phi_\ell(x_\ell) + \lambda(e + z + \sum_{\ell=1}^{L} c_\ell x_\ell) + \sum_{\ell=1}^{L} \mu_\ell(x_\ell - f_\ell(c_\ell + t_\ell)) \]

\[ + \nu \left( R - \sum_{\ell=1}^{L} t_\ell x_\ell \right). \]

1. It can happen that the constraint qualification fails to hold at the optimum. One way that this can occur is if \( R \) is the maximum amount that can be raised by any tax system.

2. When the constraint qualification holds, the necessary conditions (in ad-
dition to the constraints are:
\[ 0 = \frac{\partial L}{\partial z} = 1 - \lambda; \]
\[ 0 = \frac{\partial L}{\partial x_\ell} = \frac{d\phi_\ell}{dx_\ell}(x_\ell) - \lambda c_\ell + \mu_\ell - \nu t_\ell \quad (\ell = 1, \ldots, L); \]
\[ 0 = \frac{\partial L}{\partial t_\ell} = -\mu_\ell \frac{df_\ell}{dp_\ell}(c_\ell + t_\ell) - \nu x_\ell \quad (\ell = 1, \ldots, L). \]

3. In addition, each multiplier is nonnegative, and nonzero only when the corresponding constraint is effective.

C. Analysis

1. The first equation above implies that \( \lambda = 1 \), so that the first constraint must be satisfied with equality. Intuitively, this means simply that the optimum does not involve “throwing away” labor.

2. Substituting, the system above reduces to:
\[ 0 = \frac{d\phi_\ell}{dx_\ell}(x_\ell) - c_\ell + \mu_\ell - \nu t_\ell \quad (\ell = 1, \ldots, L); \]
\[ 0 = -\mu_\ell \frac{df_\ell}{dp_\ell}(c_\ell + t_\ell) - \nu x_\ell \quad (\ell = 1, \ldots, L). \]

3. One possibility is that \( \nu = 0 \) which would imply that \( \mu_\ell = 0 \) for all \( \ell \). (Since \( \phi_\ell \) is differentiably strictly concave, \( \frac{df_\ell}{dp_\ell} \) cannot be zero unless \( x_\ell = 0 \), meaning that good \( \ell \) has, literally, been taxed out of existence. One can easily show that this cannot happen at an optimum.) Then \( \frac{d\phi_\ell}{dx_\ell}(x_\ell) = c_\ell \) for all \( \ell \), which implies that \( t_\ell = 0 \) for all \( \ell \) and thus that total revenue tax is zero. We will ignore this case.

4. Once we know that \( \nu > 0 \), it must be the case that \( \mu_\ell > 0 \) for all \( \ell \). In turn this implies that the corresponding constraint is effective, so that \( x_\ell = f_\ell(c_\ell + t_\ell) \) for all \( \ell \), which in turn implies that \( \frac{d\phi_\ell}{dx_\ell}(x_\ell) = c_\ell + t_\ell \) for all \( \ell \).

5. Substituting this last result, the system above becomes:
\[ 0 = \mu_\ell + (1 - \nu)t_\ell \quad (\ell = 1, \ldots, L); \]
\[ 0 = -\mu_\ell \frac{df_\ell}{dp_\ell}(c_\ell + t_\ell) - \nu x_\ell \quad (\ell = 1, \ldots, L). \]
and using the first equation to eliminate $\mu_\ell$, then solving for $t_\ell$, we obtain

$$t_\ell = \frac{\nu}{1 - \nu \frac{d_\ell}{dp_\ell} (c_\ell + t_\ell)} \quad (\ell = 1, \ldots, L).$$

Recall that

$$\eta_\ell(p_\ell) := -\frac{p_\ell}{x_\ell} \frac{df_\ell}{dp_\ell}(p_\ell)$$

is the price elasticity of demand (percentage decrease in the amount demanded for a 1% increase in price). We now arrive at

$$t_\ell = \frac{\nu}{1 - \nu \eta_\ell(c_\ell + t_\ell)}.$$

V. Remarks

A. Variants of this model have been used to study transfer pricing within a firm, where such pricing might include mark-ups intended to represent the cost of some shared overhead.

B. One can extend this analysis to multi-agent economies in which the idea is to maximize the sum of the agent’s utilities.

1. The conclusion of the analysis is similar: percentage mark-ups should be inversely proportional to elasticities.

C. The extension to multiagent economies can be criticized as weighting a marginal dollar of utility for a rich person equally with a marginal dollar for a poor person.

1. This assumption is sometimes justified by suggesting that other taxes have already been used to achieve an equitable income distribution.

   a. This is unrealistic to at least some extent: the use of taxes to smooth the inequalities in the income distribution is limited by the same sort of distortions of incentives studied here.

2. When one extends to the problem of maximizing a weighted sum of utilities, there is no longer a simple conclusion.
3. In practice the goods with low price elasticities (e.g., food, alcohol, gasoline, aspirin) tend to be goods with low income elasticities, and are consumed by rich and poor in similar quantities.