Economics 5113  
Introduction to Mathematical Economics  
Winter 1999  

Lecture 13  
The Envelope Theorem  

I. Introduction  
A. The envelope theorem is a general principle describing how the value of an optimization problem changes as the parameters of the problem change.  
   1. In principle it is only a minor reduction in computational complexity.  
   2. In practice it often clarifies one’s view of a problem, and it is particularly important in allowing one to intuit the answer to certain important questions.  
B. We will begin with a general statement of the principle for unconstrained maximization.  
C. The idea will then be applied to topics in consumer theory that are often described by the term “duality.”  
D. We then go on to consider the general result for constrained problems, formally substantiating the intuition that the Lagrangean multipliers are shadow prices associated with changing the allowed value of the constraint function.  

II. The General Principle  
A. Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a $C^1$ function.  
   1. We think of $\mathbb{R}^k$ as a space of parameters, while $\mathbb{R}^n$ is the space of choice variables.
2. Thus we will consider the problem

$$\max_{z \in \mathbb{R}^n} f(z, \alpha)$$

where $\alpha \in \mathbb{R}^k$.

B. Let $x^*: U \to \mathbb{R}^n$ be a $C^1$ function defined on an open set $U \subset \mathbb{R}^k$ with the property that for each $\alpha \in U$, $x^*(\alpha)$ solves the problem above, so that $f(x^*(\alpha), \alpha) \geq f(z, \alpha)$ for all $z \in \mathbb{R}^n$.

1. Assume that $f$ is a $C^2$. Focusing on a particular point $\alpha_0 \in U$, the matrix

$$\begin{bmatrix}
\frac{\partial^2 f}{\partial z_1 \partial z_1}(x^*(\alpha_0), \alpha_0) & \cdots & \frac{\partial^2 f}{\partial z_1 \partial z_n}(x^*(\alpha_0), \alpha_0) \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial z_n \partial z_1}(x^*(\alpha_0), \alpha_0) & \cdots & \frac{\partial^2 f}{\partial z_n \partial z_n}(x^*(\alpha_0), \alpha_0)
\end{bmatrix}$$

is negative semi-definite.

a. It is nonsingular if and only if it is negative definite. In this case the implicit function theorem implies that,

2. Define $V: \mathbb{R}^n \to \mathbb{R}$ by $V(\alpha) = f(x^*(\alpha), \alpha)$.

3. Typically $V$ is called the value function since it expresses the value of the problem as a function of the given parameter.

C. The envelope theorem is the assertion that for any $\alpha \in U$ and any $i = 1, \ldots, k$,

$$\frac{\partial V}{\partial \alpha_i}(\alpha) = \frac{\partial f}{\partial \alpha_i}(\alpha, x^*(\alpha)).$$

1. The proof is very simple: applying the chain rule yields

$$\frac{\partial V}{\partial \alpha_i}(\alpha) = \frac{\partial f}{\partial \alpha_i}(x^*(\alpha), \alpha) + \sum_{j=1}^n \frac{\partial f}{\partial z_j}(x^*(\alpha), \alpha) \cdot \frac{\partial x_j}{\partial \alpha_i}(\alpha),$$

and the first order conditions for optimization imply that

$$\frac{\partial f}{\partial z_j}(x^*(\alpha), \alpha) = 0 \quad (j = 1, \ldots, k).$$
D. To develop some appreciation for this principle, suppose you are a management consultant who has been asked to predict how a small increase in the cost of one of a firm’s inputs will impact on the firm’s profits.

1. Initially it might seem that this is a very complicated problem, since, in response to the change in its input price, the firm may choose to change its input bundle. One might think that the only way to approach this problem is to look at the firm’s production function, determine the new optimal bundle of inputs, and compute profits in the new situation.

2. When you know the envelope theorem, your problem is very simple. You just determine how much of the input the firm is currently using, then multiply that amount by the price change.

3. If the change in price is large, then the envelope theorem is not directly applicable, in the sense that the derivative of the profit function is not a reliable guide to the total change in profit. Still, the underlying intuition is useful, in that the idea is that at worst the firm could simply continue with its current practices, so that the price change multiplied by the current input usage is an upper bound on the reduction in profits.

III. Duality in the Theory of the Consumer

A. We now treat some derived functions that emerge from natural problems in the theory of the consumer, and which involve interesting applications of the envelope theorem.

1. These functions arise naturally in welfare economics.

2. They have also found important applications in the econometric theory, since they are useful foundations for the types of statistical models that are applied there.

B. Let $u : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ be a consumer’s utility function.

1. We will always assume that it is $C^1$ and strictly increasing, in the sense
that the partials $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are everywhere positive.

2. In what follows we will also assume that the derived functions we discuss are $C^1$, but you should be warned that the question of what assumptions on $u$ justify this is quite complicated.

3. The derived functions of interest are as follows:

- $v(p_x, p_y, I) = \max_{p_x, x + p_y y \leq I} u(x, y)$ — the indirect utility function
- $(x(p_x, p_y, I), y(p_x, p_y, I)) = \arg\max_{p_x, x + p_y y \leq I} u(x, y)$ — the Marshallian demand functions
- $e(p_x, p_y, v) = \min_{u(x,y) \geq v} p_x x + p_y y$ — the expenditure function
- $(h(p_x, p_y, v), k(p_x, p_y, v)) = \arg\min_{u(x,y) \geq v} p_x x + p_y y$ — the Hicksian (compensated) demand functions

4. We now have the following consequences of the envelope theorem:

**Proposition 1:**

(a) $\frac{\partial v}{\partial p_x}(p_x, p_y, I) = x(p_x, p_y, I) \cdot \frac{\partial v}{\partial I}(p_x, p_y, I)$;

(b) $\frac{\partial v}{\partial p_y}(p_x, p_y, I) = y(p_x, p_y, I) \cdot \frac{\partial v}{\partial I}(p_x, p_y, I)$.

**Proof:** To achieve the framework of the envelope theorem (unconstrained optimization) we use the constraint to eliminate one variable, so that

$v(p_x, p_y, I) = \max u(x, \frac{I - p_x x}{p_y}).$

Applying the envelope theorem yields

$$\frac{\partial v}{\partial p_x} = -\frac{x}{p_y} \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial I} = -\frac{1}{p_y} \frac{\partial u}{\partial y},$$

which yields (a) when $x = x(p_x, p_y, I)$. The proof of (b) is similar. 

**Proposition 2:**

(a) $\frac{\partial e}{\partial p_x}(p_x, p_y, v) = h(p_x, p_y, v)$;
(a) \( \frac{\partial x}{\partial p_y}(p_x, p_y, v) = k(p_x, p_y, v) \).

**Proof:** In this case the constraint \( u(x, y) \geq v \) does not allow one to solve for \( y \) explicitly. We use the implicit function theorem to define a function \( f \) such that \( u(x, f(x, v)) = v \). Then

\[
e(p_x, p_y, v) = \min p_x x + p_y f(x, v)
\]

and the envelope theorem yields (a) when \( x = h(p_x, p_y, v) \). The envelope theorem also implies (b) once we realize that \( k(p_x, p_y, v) = f(h(p_x, p_y, v), v) \).  

IV. The Envelope Theorem for Constrained Optimization

A. Suppose now that we are given

\[
f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m.
\]

1. We are studying the problem of choosing \( z \) to maximize \( f(z, \alpha) \) subject to the constraint \( g(z) = 0 \).

2. Let \( \mathcal{L} : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^m \) be the Lagrangean function for this problem:

\[
\mathcal{L}(z, \alpha, \lambda) := f(z, \alpha) + \lambda \cdot g(z, \alpha).
\]

B. Suppose that \( U \subset \mathbb{R}^k \) is open, and that \( x^* : U \to \mathbb{R}^n \) and \( \lambda^* : U \to \mathbb{R}^m \) satisfy the first order conditions for optimization:

\[
\frac{\partial \mathcal{L}}{\partial z_i}(x^*(\alpha), \alpha, \lambda^*(\alpha)) = 0 \quad \text{for} \ i = 1, \ldots, n, \ \text{and}
\]

\[
\frac{\partial \mathcal{L}}{\partial \lambda_j}(x^*(\alpha), \alpha, \lambda^*(\alpha)) = 0 \quad \text{for} \ j = 1, \ldots, m.
\]

1. We have not, in fact, assumed that \( x^*(\alpha) \) solves the maximization problem, but of course in almost all applications this will be the case. Define the *value function* to be the function \( V : U \to \mathbb{R} \) defined by

\[
V(\alpha) = f(x^*(\alpha), \alpha).
\]
**Theorem:** Under the assumptions described above,

\[ \frac{\partial V}{\partial \alpha_h}(\alpha) = \frac{\partial f}{\partial \alpha_h}(x^*(\alpha), \alpha) + \sum_{j=1}^{m} \lambda^*_j(\alpha) \frac{\partial g_j}{\partial \alpha_h}(x^*(\alpha), \alpha). \]

**Proof:** We have the following computation in which, after a certain point, we do not write the points at which the various functions are evaluated, since they may be inferred:

\[
\frac{\partial V}{\partial \alpha_h} = \frac{\partial}{\partial \alpha_h} [\mathcal{L}(x^*(\alpha), \alpha, \lambda^*(\alpha))] \\
= \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \cdot \frac{\partial x_i^*}{\partial \alpha_h} + \sum_{j=1}^{m} \left( \frac{\partial \lambda^*_j}{\partial \alpha_h} g_j + \lambda^*_j \cdot \left( \sum_{i=1}^{n} \frac{\partial g_j}{\partial z_i} + \frac{\partial g_j}{\partial \alpha_h} \right) \right) \\
= \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \cdot \frac{\partial x_i^*}{\partial \alpha_h} + \sum_{j=1}^{m} \left( \lambda^*_j \cdot \frac{\partial x_i^*}{\partial \alpha_h} + \frac{\partial f}{\partial \alpha_h} \right) + \sum_{j=1}^{m} \lambda^*_j \cdot \frac{\partial g_j}{\partial \alpha_h} \\
= \frac{\partial f}{\partial \alpha_h} + \sum_{j=1}^{m} \lambda^*_j \cdot \frac{\partial g_j}{\partial \alpha_h}. \]

C. With this result, the facts about the indirect utility function and the expenditure function follow directly.

1. For the optimization problem defining the indirect utility function

\[ v(p_x, p_y, I) := \max_{x, y : x + p_y y = I} u(x, y) \]

we have \( f(x, y, p_x, p_y, I) = u(x, y) \) and \( g(x, y, p_x, p_y, I) = I - p_x x - p_y y \). From the Theorem we get

\[ \frac{\partial v}{\partial p_x}(p_x, p_y, I) = \lambda^*(p_x, p_y, I) x^*(p_x, p_y, I) \quad \text{and} \]

\[ \frac{\partial v}{\partial I}(p_x, p_y, I) = \lambda^*(p_x, p_y, I). \]

Assuming that the partials of \( u \) are everywhere positive, the Lagrangean conditions imply that \( \lambda^* \neq 0 \), after which the equation \( \frac{\partial v}{\partial p_x} = x^* \frac{\partial v}{\partial I} \) follows from division.
2. For the optimization problem defining the expenditure function

\[ e(p_x, p_y, v) = \min_{u(x,y) \geq v} p_x x + p_y y \]

we have \( f(x, y, p_x, p_y, v) = p_x x + p_y y \) and \( g(x, y, p_x, p_y, v) = v - u(x, y) \).

The equation \( \frac{\partial e}{\partial p_x} = h(p_x, p_y, v) \) now follows from the theorem simply because \( p_x \) does not affect \( g \).