Economics 5113
Introduction to Mathematical Economics
Winter 1999

Lecture 9
The Consumer, the Firm, and an Economy

I. Introduction

A. The material discussed here emphasizes how certain basic concepts are correctly understood as derivatives or partial derivatives.

B. The focus is on the central behavioral models in economics.

   1. The consumer’s choice of a consumption bundle from the set of bundles satisfying a budget constraint.

   2. A single-product firm’s choice of an input bundle, given input wages and the price of output.

C. In addition we consider Edgeworth box models of exchange (no production), considering notions of efficiency, perfectly competitive equilibrium, and fairness.

D. Since there is no one section in Novshek that covers this material, I am distributing part of a chapter from *Introductory Mathematical Economics* by D. Wade Hands.

II. The Theory of the Consumer

A. We consider the problem of utility maximization in

\[ \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0 \}. \]

   1. Let \( U : \mathbb{R}_+^n \to \mathbb{R} \) be the consumer’s utility function.
a. We will typically not fuss with assumptions about which partial derivatives exist, adopting the attitude that whenever we refer to a derivative, it has implicitly been assumed to exist.

b. The main example we consider is the Cobb-Douglas utility function

\[ U(x) = x_1^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}}. \]

2. The budget constraint is \( p_1 x_1 + \cdots + p_n x_n \leq M \), where \( p_1, \ldots, p_n > 0 \) are the prices of the goods and \( M > 0 \) is the consumer’s total income.

B. We express the economically relevant concepts in terms of partial derivatives.

1. At a particular bundle \( x \), the marginal utility of good \( i \) is the partial derivative of \( U \) with respect to \( x_i \). We denote this by \( MU_i(x), U_i(x) \), or \( \frac{\partial U}{\partial x_i}(x) \).

   a. for the Cobb-Douglas example we have

   \[
   \frac{\partial U}{\partial x_i}(x) = \frac{1}{n} \cdot x_1^{\frac{1}{n}} \cdot \cdots \cdot x_{i-1}^{\frac{1}{n}} \cdot x_i^{\frac{1}{n}-1} \cdot x_{i+1}^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}}.
   \]

2. Both in a commonsensical emotional sense, and also in the sense that people are typically risk averse, it is natural to suppose that the marginal utility of any good declines as the consumption of that good increases, holding the consumption of other goods fixed. This property is known as diminishing marginal utility.

   a. This can be expressed by the inequality \( \frac{\partial^2 U}{\partial x_i^2}(x) < 0 \).

   b. For the Cobb-Douglas utility function we have

   \[
   \frac{\partial^2 U}{\partial x_i^2}(x) = \frac{1}{n} \cdot (\frac{1}{n} - 1) \cdot x_1^{\frac{n}{n}} \cdot \cdots \cdot x_{i-1}^{\frac{1}{n}} \cdot x_i^{\frac{1}{n}-2} \cdot x_{i+1}^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}} < 0.
   \]

c. It is typically more readable to write \( U_{ij}(x) \) in place of \( \frac{\partial^2 U}{\partial x_i \partial x_j}(x) \).

d. If \( i \neq j \), and \( U_{ij}(x) > 0 \), then we say that good \( i \) and good \( j \) are complements, meaning that an increase in the quantity of good \( i \)
increases the marginal utility of good \( j \), and they are substitutes if \( U_{ij}(x) < 0 \).

i. It is important to recognize that this notion refers to a specific utility function at a specific point.

ii. On the other hand Young’s theorem (if all second partials exist and are continuous, then \( U_{ij} = U_{ji} \) for all \( i, j = 1, \ldots, n \)) implies that it makes no difference whether we think of good \( i \) as a complement for good \( j \) or vice versa.

3. The consumer’s willingness to accept decreased consumption of good \( j \) in return for increased consumption of good \( i \), at a particular point \( x \), is captured by the marginal rate of substitution between these two goods at \( x \).

a. Formally we write

\[
MRS_{ij}(x) = -\frac{dx_j}{dx_i} \bigg|_{U(x') = U(x)},
\]

where \( \frac{dx_j}{dx_i} \bigg|_{U(x') = U(x)} \) is the ‘slope’ of the curve through \( x \) given by the conditions \( x'_k = x_k \), \( k \neq i, j \), and \( U(x') = U(x) \).

b. We project this curve onto the \((x_i, x_j)\)-plane, regard it as the graph of a relation expressing \( x_j \) as a function of \( x_i \), and apply Taylor’s theorem, obtaining \( MRS_{ij}(x) = \frac{U_i(x)}{U_j(x)} \).

c. For the Cobb-Douglas case we have \( MRS_{ij}(x) = \frac{x_j}{x_i} \).

4. We now suppose that \( x \) maximizes utility subject to the budget constraint, and derive calculus-based necessary conditions.

a. Consider the curve through \( x \) given by the conditions \( x'_k = x_k \), \( k \neq i, j \), and \( p_1 x_1 + \ldots + p_n x_n = p_1 x'_1 + \ldots + p_n x'_n \).

b. If we project this curve onto the \((x_i, x_j)\)-plane, regard it as the graph of a relation expressing \( x_j \) as a function of \( x_i \), and apply
Taylor’s theorem to the condition that utility cannot be increasing in either direction along this curve, we obtain \( \frac{\partial U_i(x)}{\partial x_i} = \frac{U_i(x)}{p_i} \), or \( \frac{U_i(x)}{p_i} = \frac{U_j(x)}{p_j} \).

5. In the Cobb-Douglas case this gives \( \frac{x_i}{x_j} = \frac{p_i}{p_j} \) or \( p_1x_1 = \ldots = p_nx_n \).

The only point satisfying this condition and the budget constraint is \( x = (\frac{M}{np_1}, \ldots, \frac{M}{np_n}) \).

III. The Theory of the Firm

A. We consider the problem of choosing a profit maximizing bundle of inputs in

\[ \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0 \} \]

1. Let \( f : \mathbb{R}_+^n \rightarrow \mathbb{R} \) be the firm’s production function.

   a. As above, the main example we consider is the Cobb-Douglas production function

   \[ f(x) = x_1^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}} \]

   a. Note that \( f \) exhibits constant returns to scale: for any input bundle \( x \) and any \( \alpha > 0 \), \( f(\alpha x) = \alpha f(x) \).

2. The profit function is \( pf(x) - (w_1x_1 + \ldots + w_nx_n) \), where \( p \) is the price of output and \( w_1, \ldots, w_n > 0 \) are the wages of the factors of production.

B. We express the economically relevant concepts in terms of partial derivatives.

1. At a particular input bundle \( x \), the marginal product of input \( i \) is the partial derivative of \( f \) with respect to \( x_i \). We denote this by \( \text{MP}_i(x) \), \( f_i(x) \), or \( \frac{\partial f}{\partial x_i}(x) \).
a. For the Cobb-Douglas example we have
\[
\frac{\partial f}{\partial x_i}(x) = \frac{1}{n} \cdot x_1^{\frac{1}{n}} \cdot \ldots \cdot x_i^{\frac{1}{n}} \cdot \frac{1}{n-1} \cdot x_{i+1}^{\frac{1}{n}} \cdot \ldots \cdot x_n^{\frac{1}{n}}.
\]

2. It is typically the case in most production processes that the marginal product of each factor declines as the amount of that input increases, holding the amounts of other inputs fixed. This property is known as diminishing marginal productivity or diminishing returns.
   a. This can be expressed by the inequality \( \frac{\partial^2 f}{\partial x_i^2}(x) < 0 \).
   b. For the Cobb-Douglas utility function we have
\[
\frac{\partial^2 f}{\partial x_i^2}(x) = \frac{1}{n} \cdot \left( \frac{1}{n} - 1 \right) \cdot x_1^{\frac{1}{n}} \cdot \ldots \cdot x_i^{\frac{1}{n}} \cdot \frac{1}{n-2} \cdot x_{i+1}^{\frac{1}{n}} \cdot \ldots \cdot x_n^{\frac{1}{n}} < 0.
\]
   c. If \( i \neq j \), and \( f_{ij}(x) > 0 \), then we say that input \( i \) and input \( j \) are complements, meaning that an increase in the quantity of input \( i \) increases the marginal productivity of input \( j \), and they are substitutes if \( f_{ij}(x) < 0 \).

3. The production technology’s ability to compensate for a decrease in the amount of input \( i \) with an increase in the amount of factor \( j \), at a particular point \( x \), is captured by the marginal rate of technical substitution between these two inputs at \( x \).
   a. Formally we write
\[
\text{MRTS}_{ij}(x) = -\frac{dx_j}{dx_i}(x') = f(x'),
\]
where \( \frac{dx_j}{dx_i}(x') = f(x) \) is the ‘slope’ of the curve through \( x \) given by the conditions \( x'_k = x_k \), \( k \neq i, j \), and \( f(x') = f(x) \).
   b. We project this curve onto the \((x_i, x_j)\)-plane, regard it as the graph of a relation expressing \( x_j \) as a function of \( x_i \), and apply Taylor’s theorem, obtaining \( \text{MRTS}_{ij}(x) = \frac{f_i(x)}{f_j(x)} \).
c. For the Cobb-Douglas case we have \( \text{MRTS}_{ij}(x) = \frac{x_j}{x_i} \).

4. We now suppose that \( x \) maximizes profits, and derive calculus-based necessary conditions.

a. Consider the curve through \( x \) given by the conditions \( x_k = x_k, \ k \neq i, j, \) and \( w_1 x_1 + \ldots + w_n x_n = w_1 x_1' + \ldots + w_n x_n' \).

b. If we project this curve onto the \((x_i, x_j)\)-plane, regard it as the graph of a relation expressing \( x_j \) as a function of \( x_i \), and apply Taylor’s theorem to the condition that utility cannot be increasing in either direction along this curve, we obtain \( \text{MRTS}_{ij}(x) = \frac{f_i(x)}{f_j(x)} = \frac{w_i}{w_j} \), or \( \frac{f_i(x)}{w_i} = \frac{f_j(x)}{w_j} \).

c. In the Cobb-Douglas case this gives \( \frac{x_j}{x_i} = \frac{w_i}{w_j} \) or \( w_1 x_1 = \cdots = w_n x_n \).

Insofar as the technology exhibits constant returns to scale, there are three possibilities for profit maximization:

i. Average cost is above price, so that profits are maximized by producing nothing.

ii. Average cost is below price, so that profits can be made arbitrarily large by increasing the scale.

iii. Average cost equals price, in which case any cost-minimizing input bundle achieves zero profits.

5. In the two input case we can write the isoquant \( f(x_1, x_2) = \bar{f} \) as the graph of a function \( x_2 = k(x_1) \). Differentiating the identity \( f(x_1, k(x_1)) = \bar{f} \) twice and isolating the second derivative of \( k \), we find that \( k \) is convex at \( x_1 \) if, at \( x = (x_1, k(x_1)) \), we have

\[
J_f^2 f_{11} - 2 f_1 f_2 f_{12} + f_2^2 f_{22} < 0.
\]

IV. A Pure Exchange Economy

A. Many of the central concepts of economic theory are present in the model of
two-consumer, two-good pure exchange (no production) known as the Edgeworth box.

1. Let the two consumers be $A$ and $B$.
2. Let the total amounts of the two goods be $\bar{x}$ and $\bar{y}$.
3. An allocation is a pair $((x_A, y_A), (x_B, y_B)) \in (\mathbb{R}^2_+)^2$.

B. An allocation $((x_A, y_A), (x_B, y_B)) \in (\mathbb{R}^2_+)^2$ is feasible if $x_A + x_B = \bar{x}$ and $y_A + y_B = \bar{y}$.

1. The set of feasible allocations can be represented as a box in the $(x, y)$-plane, with the lower left hand corner being the point $((0, 0), (\bar{x}, \bar{y})), \text{while}$ the upper right hand corner is the point $((\bar{x}, \bar{y}), (0, 0))$.

C. Let the utility functions of the two consumers be $u_A, u_B : \mathbb{R}^2_+ \rightarrow \mathbb{R}$.

1. An allocation $((x_A, y_A), (x_B, y_B))$ is Pareto efficient if it is feasible and there is no other feasible allocation $((x'_A, y'_A), (x'_B, y'_B))$ that makes one of the consumers better off ($u_A(x'_A, y'_A) > u_A(x_A, y_A) \text{ or } u_B(x'_B, y'_B) > u_B(x_B, y_B)$) without making either worse off ($u_A(x'_A, y'_A) \geq u_A(x_A, y_A)$ and $u_B(x'_B, y'_B) \geq u_B(x_B, y_B)$).

2. Typically the set of efficient allocations is a curve in the Edgeworth box running from the lower left hand corner to the upper right hand corner.

D. We now introduce the concept of private ownership, assuming that the total allocation is originally the sum of the individual endowment bundles $(\xi_A, \omega_A)$ and $(\xi_B, \omega_B)$.

1. A Walrasian equilibrium is a price vector $p = (p_x, p_y)$, together with an allocation $((x_A, y_A), (x_B, y_B))$, satisfying the following conditions:
   a. (Optimization) $x_A, y_A$ solves the problem “max $u_A(x, y)$ subject to $p_x x + p_y y \leq p_x \xi_A + p_y \omega_A$” and $x_B, y_B$ solves the problem “max $u_B(x, y)$ subject to $p_x x + p_y y \leq p_x \xi_B + p_y \omega_B$.”
   b. (Market Clearing) $((x_A, y_A), (x_B, y_B))$ is feasible.
2. One of the most important (and deepest) theorems of economic theory asserts that, under reasonably general assumptions, for any vector of endowments there is a Walrasian equilibrium.

3. Disregarding the endowments in order to ask what allocations could result from Walrasian equilibrium relative to some endowment, we have the notion of a \textit{Walrasian allocation}, which we define to be a feasible allocation \(((x_A, y_A), (x_B, y_B))\) such that, for some price vector \((p_x, p_y)\), there is no bundle \((x, y)\) with \(u_A(x, y) > u_A(x_A, y_A)\) and \(p_xx + p_yy \leq p_xx_A + p_yy_A\), and there is also no bundle \((x, y)\) with \(u_B(x, y) > u_B(x_B, y_B)\) and \(p_xx + p_yy \leq p_xx_B + p_yy_B\).

\textbf{First Fundamental Theorem of Welfare Economics:} If \(((x_A, y_A), (x_B, y_B))\) is a Walrasian allocation, say with respect to the price vector \((p_x, p_y)\), then \(((x_A, y_A), (x_B, y_B))\) is Pareto efficient.

\textbf{Proof:} Supposing, in order to achieve a contradiction, that the claim is false, let

\[\left((x_A', y_A'), (x_B', y_B')\right)\]

be a feasible allocation that makes one of the consumers better off, say \(u_A(x_A', y_A') > u_A(x_A, y_A)\), without making the other worse off, so that \(u_B(x_B', y_B') \geq u_B(x_B, y_B)\). Since \(((x_A, y_A), (x_B, y_B))\) is Walrasian, we must have

\[p_xx'A + p_yy'A \geq p_xx_A + p_yy_A \quad \text{and} \quad p_xx'B + p_yy'B \geq p_xx_B + p_yy_B.\]

Since both \(((x_A, y_A), (x_B, y_B))\) and \(((x_A', y_A'), (x_B', y_B'))\) are feasible, we have the following
contradictory computation:

\[ px \bar{x} + py \bar{y} = px(x'_A + x'_B) + py(y'_A + y'_B) \]
\[ = (px x'_A + py y'_A) + (px x'_B + py y'_B) \]
\[ > (px x_A + py y_A) + (px x_B + py y_B) \]
\[ = px(x_A + x_B) + py(y_A + y_B) \]
\[ = px \bar{x} + py \bar{y}. \]

E. Primitive notions of equity can be expressed in this framework.

1. An allocation \(((x_A, y_A), (x_B, y_B))\) is fair (sometimes the term superfair is used) if neither agent prefers the other agent’s bundle:

\[ u_A(x_A, y_A) \geq u_A(x_B, y_B) \quad \text{and} \quad u_B(x_B, y_B) \geq u_B(x_A, y_A). \]

2. There exist allocations that are both fair and efficient. One way to obtain such an allocation is to divide the endowment vector equally between the two agents, then let them trade to a Walrasian equilibrium. (Here we are using the theorem asserting that such an equilibrium must exist.) As we have just demonstrated, the resulting allocation is necessarily efficient, and since both agents are maximizing subject to the same constraint, neither can envy the choice of the other, since he or she could also have chosen it.