Economics 5113
Introduction to Mathematical Economics
Winter 1999

Lecture 8
The Second Derivative

1. Introduction

A. Let $f : U \rightarrow \mathbb{R}$ be a function where $U \subset \mathbb{R}^n$ is open.

1. We will assume that the second partial derivative functions

$$\frac{\partial^2 f}{\partial x_i \partial x_j} : U \rightarrow \mathbb{R}$$

are defined and continuous.

2. Our goal is to develop a useful understanding of these partials.

B. In the last lecture we saw how the first order partials define a derivative

$$Df : U \rightarrow L(\mathbb{R}^n, \mathbb{R})$$

where $L(\mathbb{R}^n, \mathbb{R})$ is the space of linear functions from $\mathbb{R}^n$ to $\mathbb{R}$.

1. By analogy one might view the second derivative as a function

$$D^2 f : U \rightarrow L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}))$$

where $L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}))$ is the set of linear functions with domain $\mathbb{R}^n$ and range $L(\mathbb{R}^n, \mathbb{R})$.

2. This is possible, but not very useful.

3. The point of view we will stress is that the second partials define a quadratic approximation of $f$ that is valid “up to second order.”
II. The Symmetry of Second Partials

A. To simplify the notation we assume that the domain $U$ of $f : U \to \mathbb{R}$ is a subset of $\mathbb{R}^2$.

1. Denote the two variables by $x$ and $y$.

2. Everything below generalizes to $n$-dimensional domains without any difficulty.

**Young's Theorem:** If the second partials of $f$ are continuous, then

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$$

for all $(x, y) \in U$.

**Proof:** Fix $(x, y) \in U$, and consider numbers $\Delta x$ and $\Delta y$ such that

$$\{(x + \alpha \Delta x, y + \beta \Delta y) : 0 \leq \alpha, \beta \leq 1\} \subset U.$$

(Since $U$ is open, this inclusion will hold if $|\Delta x|$ and $|\Delta y|$ are sufficiently small.) Applying Rolle's theorem to the function

$$g(\xi) = f(\xi, y + \Delta y) - f(\xi, y),$$

we find that there is some $\xi^*$ between $x$ and $x + \Delta x$ such that

$$g(x + \Delta x) - g(x) = g'(\xi^*) \cdot \Delta x,$$

which translates to

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)$$

$$= \left[ \frac{\partial f}{\partial x}(\xi^*, y + \Delta y) - \frac{\partial f}{\partial x}(\xi^*, y) \right] \cdot \Delta x.$$

Applying Rolle's theorem to the function $h(\eta) = \frac{\partial f}{\partial x}(\xi^*, \eta)$, there is some $\eta^*$ between $y$ and $y + \Delta y$ with $h(y + \Delta y) - h(y) = h'(\eta^*) \cdot \Delta y$, which results in

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)$$

$$= \frac{\partial^2 f}{\partial x \partial y}(\xi^*, \eta^*) \cdot \Delta x \cdot \Delta y.$$
Interchanging \(x\) and \(y\) in the argument above, we can equally well find a point \((\xi^*, \eta^*)\) with

\[
f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y) = \frac{\partial^2 f}{\partial y \partial x}(\xi^*, \eta^*) \cdot \Delta x \cdot \Delta y.
\]

Thus \(\frac{\partial^2 f}{\partial x \partial y}(\xi^*, \eta^*) = \frac{\partial^2 f}{\partial y \partial x}(\xi^*, \eta^*)\). By choosing \(|\Delta x|\) and \(|\Delta y|\) small, we can force \((\xi^*, \eta^*)\) and \((\xi^{**}, \eta^{**})\) into any given neighborhood of \((x, y)\), so the claim follows from the continuity of \(\frac{\partial^2 f}{\partial x \partial y}(\cdot)\) and \(\frac{\partial^2 f}{\partial y \partial x}(\cdot)\). \(\blacksquare\)

III. The Second Derivative and Taylor’s Theorem

A. Let \(f: U \to \mathbb{R}\) be a function, where \(U \subset \mathbb{R}^n\) is open.

1. We say that \(f\) is \textit{twice differentiable} at \(x \in U\) if there is a neighborhood \(v \subset U\) of \(x\) such that \(Df(x')\) is defined at all points \(x' \in V\), and there is some quadratic form \(q\), called the \textit{second derivative} of \(f\) at \(x\), such that for any \(\epsilon > 0\) there is \(\delta > 0\) such that

\[
\left| f(x') - \left( f(x) + Df(x)(x' - x) + \frac{1}{2}q(x' - x) \right) \right| \leq \epsilon \|x' - x\|^2
\]

for all \(x' \in B_\delta(x)\).

2. When \(f\) is twice differentiable at \(x\), we let \(D^2 f(x)\) be the \(n \times n\) symmetric matrix that induces the second derivative.

B. We say that \(f\) is \(C^2\) if \(f\) is twice continuously differentiable at each \(x \in U\), and \(D^2 f(\cdot): U \to \mathbb{R}^{n \times n}\) (where \(\mathbb{R}^{n \times n}\) stands for the space of \(n \times n\) matrices) is continuous.

1. This is a little bit backwards from the usual treatment, in that here we have \textit{defined} the second derivative to be a quadratic form satisfying the conclusion of Taylor’s theorem:

\[
f(x^0) = f(x_0) + Df(x_0)(x^0 - x_0) + \frac{1}{2}(x - x_0)'D^2 f(x_0)(x - x_0) + R(x - x_0)
\]
where $R(\cdot)$ is a function, defined in a neighborhood of $0 \in \mathbb{R}^n$ that satisfies
\[
\frac{R(z)}{\|z\|^2} \to 0 \text{ as } z \to 0.
\]

2. More common is to define “$C^2$-ness” for $f$ to mean that each second partial derivative $\frac{\partial^2 f}{\partial x_i \partial x_j} : U \to \mathbb{R}$ is defined and continuous.

3. I prefer the definition here, because it expresses the desired concept directly.

4. Due to our choice of definition, Taylor’s Theorem has a different appearance:

**Taylor’s Theorem:** The following conditions are equivalent:

(a) $f$ is $C^2$;

(b) each second partial $\frac{\partial^2 f}{\partial x_i \partial x_j} : U \to \mathbb{R}$ is defined and continuous.

When these hold $D^2 f(x)$ is, for each $x \in U$, the matrix of second partials $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$.

C. Remarks on the Proof

1. The assertion consists of two implications, neither of which will be proved here. For the part asserting that $f$ is $C^2$ whenever all second partials are $C^2$, I can think of two rather unappealing methods of proof that probably work.

   a. A tricky application of Rolle’s Theorem, similar to the proof of Young’s Theorem.

   b. Redoing everything from the point of view considered, and rejected, at the beginning: the second derivative is the first derivative of the first derivative. Recall that the univariate version of Taylor’s Theorem was proved by induction, looking at the $n$th derivation of the $(n - 1)$st derivative.
IV. Second Order Conditions for Maximization

A. Necessary Conditions

Theorem: If $U \subset \mathbb{R}^n$ is open, $f: U \to \mathbb{R}$ is $C^2$, and $x^* \in U$ is a local maximum for $f$, then $Df(x^*) = 0$ and $D^2f(x^*)$ is negative semidefinite.

Proof: We have already shown that $Df(x^*) = 0$. If $D^2f(x^*)$ is not negative semidefinite, there is some $v \in \mathbb{R}^n$ with $v'D^2f(x^*)v > 0$. For $\epsilon > 0$ small, and $\alpha > 0$ small given the choice of $\epsilon$, we have

$$f(x^* + \alpha v) \geq f(x^*) + \frac{1}{2}(\alpha v)'D^2f(x^*)(\alpha v) - \epsilon \|\alpha v\|^2$$

$$= f(x^*) + \alpha^2 \left( \frac{1}{2}v'D^2f(x^*)v - \epsilon \|v\|^2 \right)$$

$$> f(x^*).$$

This is a contradiction of the assumption that $x^*$ is a local maximum. □

B. Sufficient Conditions

Theorem: If $U$ is open, $f: U \to \mathbb{R}$ is $C^2$, $x^* \in U$, $Df(x^*) = 0$, and $D^2f(x^*)$ is negative definite, then $x^*$ is a strict local maximum of $f$.

Proof: Let $S^{n-1} = \{ v \in \mathbb{R}^n : \|v\| = 1 \}$ be the unit $(n-1)$-sphere in $\mathbb{R}^n$. Clearly $S^{n-1}$ is bounded, and since $\| \cdot \|: \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $S^{n-1}$ is closed. Thus $S^{n-1}$ is compact. The function $v \to v'D^2f(x^*)v$ is continuous (only matrix multiplication is involved) so

$$\arg \max_{v \in S^{n-1}} v'D^2f(x^*)v \neq \emptyset.$$

Let $\bar{v}$ be a maximizer. Setting $\epsilon = \bar{v}'D^2f(x^*)\bar{v}/3$ there is $\delta > 0$ small enough that if $\|x - x^*\| < \delta$, and $x \neq x^*$, then

$$f(x) \leq f(x^*) + \frac{1}{2}(x - x^*)'D^2f(x^*)(x - x^*) + \epsilon \|x - x^*\|^2$$

$$= f(x^*) + \|x - x^*\|^2 \cdot \left( \frac{1}{2} \frac{x - x^*}{\|x - x^*\|} D^2f(x^*) \frac{x - x^*}{\|x - x^*\|} + \epsilon \right)$$

$$\leq f(x^*) + \|x - x^*\|^2 \cdot \left( \frac{1}{2} \bar{v}'D^2f(x^*)\bar{v} + \epsilon \right)$$

$$< f(x^*).$$ □