Lecture 6

Convexity
Agenda

There are three main topics:

- Convex sets.
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- Convex sets.
- Convexity and concavity of functions.
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- Convex sets.
- Convexity and concavity of functions.
- Applications to optimization and economics.
Objectives Related to Convex Sets

We will study:

- The definition of a convex set.
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- The inner product of two vectors.
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- The Cauchy-Schwartz inequality.
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- Sufficient conditions for convexity.
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- The definition of a convex set.
- The inner product of two vectors.
- The Cauchy-Schwartz inequality.
- Sufficient conditions for convexity.
- The separating hyperplane theorem.
Definition of a Convex Set

Definition: A set $C \subseteq \mathbb{R}^m$ is convex if

$$(1 - \alpha)x + \alpha y \in C$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1$. 

Visually, this means that the line segment between any two of the set's points is contained in the set.

A room is convex if two people in it can always see each other.
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A First Example

- Prove that \( \{ x \in \mathbb{R}^m : x_1 \geq 0 \} \) is convex.
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• Prove that \( \{ x \in \mathbb{R}^m : x_1 \geq 0 \} \) is convex.

Proof: If \( x, y \in \mathbb{R}^m \) and \( 0 \leq \alpha \leq 1 \), then

\[
(1-\alpha)x + \alpha y = ((1-\alpha)x_1 + \alpha y_1, \ldots, (1-\alpha)x_m + \alpha y_m).
\]

If \( x_1 \geq 0 \) and \( y_1 \geq 0 \), then \( (1 - \alpha)x_1 + \alpha y_1 \geq 0 \) because sums and products of nonnegative numbers are nonnegative.
The Inner Product

**Definition:** The inner product of two vectors $x, y \in \mathbb{R}^m$ is

$$\langle x, y \rangle := \sum_{i=1}^{m} x_i y_i.$$
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• For $x, y, z \in \mathbb{R}^m$ and $t \in \mathbb{R}$ prove that:
  • $\langle x, y \rangle = \langle y, x \rangle$;
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- For \( x, y, z \in \mathbb{R}^m \) and \( t \in \mathbb{R} \) prove that:
  - \( \langle x, y \rangle = \langle y, x \rangle \);
  - \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \);
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- For $x, y, z \in \mathbb{R}^m$ and $t \in \mathbb{R}$ prove that:
  - $\langle x, y \rangle = \langle y, x \rangle$;
  - $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
  - $\langle tx, y \rangle = t \langle x, y \rangle$. 
A More General Example

- For $p \in \mathbb{R}^m$ and $c \in \mathbb{R}$ prove that
  \[ \{ x \in \mathbb{R}^m : \langle p, x \rangle \geq c \} \]
  is convex.
A More General Example

- For \( p \in \mathbb{R}^m \) and \( c \in \mathbb{R} \) prove that

\[
\{ x \in \mathbb{R}^m : \langle p, x \rangle \geq c \}
\]

is convex.

**Proof:** Suppose that \( x, y \in \mathbb{R}^m, \langle p, x \rangle \geq c, \langle p, y \rangle \geq c, \) and \( 0 \leq \alpha \leq 1. \) Then

\[
\langle p, (1 - \alpha)x + \alpha y \rangle = (1 - \alpha)\langle p, x \rangle + \alpha\langle p, y \rangle \\
\geq (1 - \alpha)c + \alpha c = c.
\]
Length and Orthogonality

Definition: The length of a vector $x \in \mathbb{R}^m$ is

$$\|x\| := \sqrt{\langle x, x \rangle}.$$
Length and Orthogonality

**Definition**: The length of a vector $x \in \mathbb{R}^m$ is

$$\|x\| := \sqrt{\langle x, x \rangle}.$$ 

**Definition**: Two vectors $x, y \in \mathbb{R}^m$ are perpendicular or orthogonal if $\langle x, y \rangle = 0$. We write $x \perp y$ to indicate that this is the case.
The Pythagorean Theorem

- Prove that if \((y - x) \perp (z - x)\), then

\[
\|y - z\|^2 = \|y - x\|^2 + \|z - x\|^2.
\]

(Hint: \(y - z = (y - x) - (z - x)\)).
Proof:

\[
\|y - z\|^2 = \|(y - x) - (z - x)\|^2 \\
= \langle(y - x) - (z - x), (y - x) - (z - x)\rangle \\
= \langle y - x, y - x \rangle + \langle y - x, z - x \rangle \\
\quad + \langle z - x, y - x \rangle + \langle z - x, z - x \rangle \\
= \langle y - x, y - x \rangle + \langle z - x, z - x \rangle \\
= \|y - z\|^2 + \|y - z\|^2.
\]
Cauchy-Schwartz Inequality

- For given $x, y \in \mathbb{R}^m$, find the value of $\alpha$ that minimizes

$$
\|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y \rangle
$$

$$
= \langle x, x \rangle + 2\alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle.
$$
Cauchy-Schwartz Inequality

- For given $x, y \in \mathbb{R}^m$, find the value of $\alpha$ that minimizes

$$\|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y \rangle$$

$$= \langle x, x \rangle + 2\alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle.$$ 

$$\alpha^* = -\frac{\langle x, y \rangle}{\langle y, y \rangle}.$$
• Use the fact that $0 \leq \|x + \alpha^*y\|^2$ to derive an inequality relating $\langle x, x \rangle$, $\langle x, y \rangle$, and $\langle y, y \rangle$. This is the Cauchy-Schwartz inequality. It is extremely important and useful.
Use the fact that \( 0 \leq \| x + \alpha^* y \|^2 \) to derive an inequality relating \( \langle x, x \rangle \), \( \langle x, y \rangle \), and \( \langle y, y \rangle \).

\[
0 \leq \langle x, x \rangle - 2 \frac{\langle x, y \rangle}{\langle y, y \rangle} \langle x, y \rangle + \frac{\langle x, y \rangle^2}{\langle y, y \rangle^2} \langle y, y \rangle.
\]

\[\Rightarrow \langle x, y \rangle \leq \| x \| \cdot \| y \| .\]
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0 \leq \langle x, x \rangle - 2\frac{\langle x, y \rangle}{\langle y, y \rangle}\langle x, y \rangle + \frac{\langle x, y \rangle^2}{\langle y, y \rangle^2}\langle y, y \rangle.
\]

\[
\Rightarrow \langle x, y \rangle \leq \|x\| \cdot \|y\|.
\]

• This is the **Cauchy-Schwartz inequality**. It is extremely important and useful.

• In fact $\langle x, y \rangle = \|x\| \cdot \|y\| \cos \angle(x, y)$. 
The Triangle Inequality

• Prove the triangle inequality: if $x, y \in \mathbb{R}^m$, then
  $$\|x + y\| \leq \|x\| + \|y\|.$$
The Triangle Inequality

- Prove the **triangle inequality**: if \( x, y \in \mathbb{R}^m \), then

\[ \| x + y \| \leq \| x \| + \| y \|. \]

**Proof:**

\[ \| x + y \| ^2 = \langle x + y, x + y \rangle \]

\[ = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \]

\[ = \| x \| ^2 + 2 \langle x, y \rangle + \| y \| ^2 \]

\[ \leq \| x \| ^2 + 2 \| x \| \cdot \| y \| + \| y \| ^2 = (\| x \| + \| y \| ) ^2. \]
Balls are Convex

- Prove that the unit ball

\[ \{ x \in \mathbb{R}^m : \|x\| \leq 1 \} \]

is convex.
Proof: Suppose that \( x, y \in \mathbb{R}^m \) with \( \|x\| \leq 1 \) and \( \|y\| \leq 1 \), and \( 0 \leq \alpha \leq 1 \). Then

\[
\langle (1 - \alpha)x + \alpha y, (1 - \alpha)x + \alpha y \rangle
\]

\[
= (1 - \alpha)^2 \langle x, x \rangle + 2\alpha(1 - \alpha) \langle x, y \rangle + \alpha^2 \langle y, y \rangle
\]

\[
= (1 - \alpha)^2 \|x\|^2 + 2\alpha(1 - \alpha) \|x\| \cdot \|y\| + \alpha^2 \|y\|^2
\]

\[
\leq (1 - \alpha)^2 \|x\|^2 + 2\alpha(1 - \alpha) \|x\| \cdot \|y\| + \alpha^2 \|y\|^2
\]

\[
= ((1 - \alpha)\|x\| + \alpha \|y\|)^2 \leq 1.
\]
Sums of Convex Sets

- Prove that if $C$ and $C''$ are convex, then

$$C + C'' := \{ x + x' : x \in C \text{ and } x' \in C'' \}$$

is convex.
Proof: Suppose that \( \bar{x}, \bar{y} \in C + C' \), and \( 0 \leq \alpha \leq 1 \). Then \( \bar{x} = x + x' \) and \( \bar{y} = y + y' \) for some \( x, y \in C \) and \( x', y' \in C' \), and

\[
(1 - \alpha)x + \alpha y \in C \quad \text{and} \quad (1 - \alpha)x' + \alpha y' \in C'
\]

because \( C \) and \( C' \) are convex. Therefore

\[
(1 - \alpha)\bar{x} + \alpha \bar{y} = (1 - \alpha)(x + x') + \alpha(y + y')
\]

\[
= ((1 - \alpha)x + \alpha y) + ((1 - \alpha)x' + \alpha y') \in C + C'.
\]
Intersecting Convex Sets

- Prove that if $I$ is any set and, for each $i \in I$, $C_i$ is convex, then $\bigcap_{i \in I} C_i$ is convex.
Intersecting Convex Sets

- Prove that if \( I \) is any set and, for each \( i \in I \), \( C_i \) is convex, then \( \bigcap_{i \in I} C_i \) is convex.

Proof: If \( x, y \in \bigcap_{i \in I} \) and \( 0 \leq \alpha \leq 1 \), then 
\[
(1 - \alpha)x + \alpha y \in C_i \text{ for each } i \text{ since } C_i \text{ is convex, so}
\]
\[
(1 - \alpha)x + \alpha y \in \bigcap_{i \in I} C_i.
\]
The Convex Hull of a Set

**Definition:** The convex hull of a set $S \subset \mathbb{R}^m$ is the smallest convex set that contains $S$. 

Proof: The intersection of all of the convex sets that contain $S$ is a subset of any convex set containing $S$, and the last result implies that it is convex.
The Convex Hull of a Set

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**Interior Points**

**Definition:** A point $x$ is an *interior point* of a set $S \subset \mathbb{R}^m$ if, for some $\varepsilon > 0$, $S$ contains the $\varepsilon$-ball around $x$:

$$\{ y \in \mathbb{R}^m : \| y - x \| < \varepsilon \} \subset S.$$
Interior Points

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- A set $U \subset \mathbb{R}^m$ is open if all of its points are interior points.
Interior Points

**Definition:** A point $x$ is an **interior point** of a set $S \subset \mathbb{R}^m$ if, for some $\varepsilon > 0$, $S$ contains the $\varepsilon$-ball around $x$:

$$\left\{ y \in \mathbb{R}^m : \|y - x\| < \varepsilon \right\} \subset S.$$

- A set $U \subset \mathbb{R}^m$ is **open** if all of its points are interior points.
- A set $K \subset \mathbb{R}^m$ is **closed** if its complement $\mathbb{R}^m \setminus K$ is open.
The Open Ball is Open

- Prove that the open unit ball
  \[ B = \{ x \in \mathbb{R}^m : \| x \| < 1 \} \]
  is, in fact, open.

Proof:
Let \( x \) be an arbitrary point of \( B \). For \( y \in \mathbb{R}^m \) the triangle inequality gives
\[ k y k = k x + (y - x) k < k x k + k y - x k. \]

Therefore, for \( y \in \mathbb{R}^m : k y - x k < 1 - k x k \) is contained in \( B \), so \( x \) is an interior point of \( B \).
The Open Ball is Open

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Proof: Let \( x \) be an arbitrary point of \( B \). For \( y \in \mathbb{R}^m \) the triangle inequality gives

\[ \|y\| = \|x + (y - x)\| \leq \|x\| + \|y - x\|. \]

Therefore \( \{ y \in \mathbb{R}^m : \|y - x\| < 1 - \|x\| \} \) is contained in \( B \), so \( x \) is an interior point of \( B \).
Separating Hyperplanes

Separating Hyperplane Theorem: If $C$ and $D$ are convex, $C$ has an interior point, and none of the interior points of $C$ are contained in $D$, then $C$ and $D$ can be separated by a hyperplane: there is $p \in \mathbb{R}^m$ and a number $c$ such that

$$C \subset \{ x \in \mathbb{R}^m : \langle p, x \rangle \leq c \}$$

and

$$D \subset \{ x \in \mathbb{R}^m : \langle p, x \rangle \geq c \}. $$
Convex Functions: Goals

- The definition of a convex (concave) function.
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- Characterization of convexity in terms of the set above the function’s graph.
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Convex Functions: Goals

- The definition of a convex (concave) function.
- Characterization of convexity in terms of the set above the function’s graph.
- The definition of a quasiconvex (quasiconcave) function.
- Characterization of quasiconvexity in terms of lower contour sets.
- Properties of the expenditure function and the indirect utility function.
Convex Functions Defined

Definition: Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is convex if

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1.$
Convex Functions Defined

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is **convex** if

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1$.

- **Definition:** A function $f : C \rightarrow \mathbb{R}$ is **concave** if $-f$ is convex. Every result concerning convex functions implies a result about concave functions, and vice versa.
Remark: In order for the definition to make sense, the domain $C$ of $f$ must be convex.
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- Let $f : [0, \infty) \rightarrow \mathbb{R}$ be the function $f(t) = -\sqrt{t}$. Then $f$ is convex, but there is no convex function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g(t) = f(t)$ for all $t \geq 0$. 
Convexity Characterized

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \rightarrow \mathbb{R}$ be a convex function. Prove that $\mathcal{F} := \{ (x, v) \in C \times \mathbb{R} : f(x) \leq v \}$ is convex.
Convexity Characterized

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be a convex function. Prove that $\mathcal{F} := \{(x, v) \in C \times \mathbb{R} : f(x) \leq v\}$ is convex.

Proof: Suppose that $(x, v), (y, w) \in \mathcal{F}$ and $0 \leq \alpha \leq 1$. Then $(1 - \alpha)(x, v) + \alpha(y, w) \in \mathcal{F}$ because

$$
(1 - \alpha)v + \alpha w \geq (1 - \alpha)f(x) + \alpha f(y)
\geq f((1 - \alpha)x + \alpha y).
$$
The Converse

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \rightarrow \mathbb{R}$ be a function. Suppose that
\[
\mathcal{F} := \{ (x, v) \in C \times \mathbb{R} : f(x) \leq v \}
\]
is convex. Prove that $f$ is convex.
The Converse

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be a function. Suppose that $\mathcal{F} := \{(x, v) \in C \times \mathbb{R} : f(x) \leq v\}$ is convex. Prove that $f$ is convex.

**Proof:** If $x, y \in C$ and $0 \leq \alpha \leq 1$, then

$$
(1 - \alpha)(x, f(x)) + \alpha(y, f(y)) \in \mathcal{F},
$$

because $\mathcal{F}$ is convex, so

$$(1 - \alpha)f(x) + \alpha f(y) \geq f((1 - \alpha)x + \alpha y).$$
The Expenditure Function

- Prove the concavity of the expenditure function

\[ E(p, u) := \min_{U(x) \geq u} p \cdot x. \]
Proof: Suppose that $p$, $q$ are price vectors and $0 \leq \alpha \leq 1$. Choose $x$ solving

$$\min_{U(x) \geq u} ((1 - \alpha)p + \alpha q) \cdot x.$$  

Then $U(x) \geq u$, so

$$(1 - \alpha)E(p, u) + \alpha E(q, u) \leq (1 - \alpha)p \cdot x + \alpha q \cdot x$$

$$= E((1 - \alpha)p + \alpha q, u).$$
**Quasiconvex Functions**

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is **quasiconvex** if

$$f((1 - \alpha)x + \alpha y) \leq \max\{f(x), f(y)\}$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1$. 
Quasiconvex Functions

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$$f((1 - \alpha)x + \alpha y) \leq \max\{f(x), f(y)\}$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1$.

- Since $(1 - \alpha)f(x) + \alpha f(y) \leq \max\{f(x), f(y)\}$, a convex function is necessarily quasiconvex.
Quasiconvex Functions

Definition: Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is quasiconvex if

$$f((1 - \alpha)x + \alpha y) \leq \max\{f(x), f(y)\}$$

whenever $x, y \in C$ and $0 \leq \alpha \leq 1$.

- Since $(1 - \alpha)f(x) + \alpha f(y) \leq \max\{f(x), f(y)\}$, a convex function is necessarily quasiconvex.

- Definition: A function $f : C \rightarrow \mathbb{R}$ is quasiconcave if $-f$ is quasiconvex.
Characterization

- Let $C \subset \mathbb{R}^m$ be quasiconvex, and let $f : C \to \mathbb{R}$ be a convex function. Prove that for any $v \in \mathbb{R}$, the lower contour set $A(v) := \{ x \in C \times \mathbb{R} : f(x) \leq v \}$ is convex.
Characterization

- Let $C \subset \mathbb{R}^m$ be quasiconvex, and let $f : C \to \mathbb{R}$ be a convex function. Prove that for any $v \in \mathbb{R}$, the lower contour set $A(v) := \{ x \in C \times \mathbb{R} : f(x) \leq v \}$ is convex.

**Proof:** Suppose that $x, y \in A(v)$ and $0 \leq \alpha \leq 1$. Then $(1 - \alpha)x + \alpha y \in A(v)$ because

$$f((1 - \alpha)x + \alpha y) \leq \max\{f(x), f(y)\} \leq v.$$
The Converse

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be a function. Suppose that for all $v \in \mathbb{R}$, $A(v)$ is convex. Prove that $f$ is quasiconvex.
The Converse

Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be a function. Suppose that for all $v \in \mathbb{R}$, $A(v)$ is convex. Prove that $f$ is quasiconvex.

Proof: If $x, y \in C$ and $0 \leq \alpha \leq 1$, then

$$(1 - \alpha)x + \alpha y \in A(\max\{f(x), f(y)\})$$

since this set contains both $x$ and $y$, but this is precisely what we need to show.
The Indirect Utility Function

- Prove the quasiconvexity of the indirect utility function

\[ V(p, I) := \max_{p \cdot x \leq I} U(x). \]
The Indirect Utility Function

- Prove the quasiconvexity of the indirect utility function

\[ V(p, I) := \max_{p \cdot x \leq I} U(x). \]

Proof: Given \((p_0, I_0), (p_1, I_1),\) and \(0 \leq \alpha \leq 1,\) let

\[ (p_\alpha, I_\alpha) := (1 - \alpha)(p_0, I_0) + \alpha(p_1, I_1). \]

Choose \(x\) solving \(\max_{p_\alpha \cdot x \leq I_\alpha} U(x).\)
Then

\[(1 - \alpha)p_0 \cdot x + \alpha p_1 \cdot x = p_\alpha \cdot x \leq I_\alpha = (1 - \alpha)I_0 + \alpha I_1,\]

so either \(p_0 \cdot x \leq I_0\) or \(p_1 \cdot x \leq I_1\). In the first case

\[V(p_\alpha, I_\alpha) \leq V(p_0, I_0)\]

and in the second case

\[V(p_\alpha, I_\alpha) \leq V(p_1, I_1),\]

so

\[V(p_\alpha, I_\alpha) \leq \max\{V(p_0, I_0), V(p_1, I_1)\}.\]
The Set of Minimizers

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be quasiconvex. Suppose $m$ is the minimum value attained by $f$. Prove that $\text{argmin } f := f^{-1}(m)$ is convex.
The Set of Minimizers

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \to \mathbb{R}$ be quasiconvex. Suppose $m$ is the minimum value attained by $f$. Prove that $\text{argmin } f := f^{-1}(m)$ is convex.

**Proof:** If $x, y \in \text{argmin } f$ and $0 \leq \alpha \leq 1$, then

$$m = \max\{f(x), f(y)\} \geq f((1 - \alpha)x + \alpha y) \geq m,$$

so $(1 - \alpha)x + \alpha y \in \text{argmin } f$. 
Strictly Convex Functions

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is strictly convex if

$$f((1 - \alpha)x + \alpha y) < (1 - \alpha)f(x) + \alpha f(y)$$

whenever $x, y \in C$, $x \neq y$, and $0 < \alpha < 1$. 

There is a necessary and sufficient condition for this in terms of $F$ (as defined above) that may figure in homework or exams.
Strictly Convex Functions

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is **strictly convex** if

$$f((1 - \alpha)x + \alpha y) < (1 - \alpha)f(x) + \alpha f(y)$$

whenever $x, y \in C$, $x \neq y$, and $0 < \alpha < 1$.

- There is a necessary and sufficient condition for this in terms of $\mathcal{F}$ (as defined above) that may figure in homework or exams.
Strict Quasiconvexity

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \rightarrow \mathbb{R}$ is **strictly quasiconvex** if

$$f((1 - \alpha)x + \alpha y) < \max\{f(x), f(y)\}$$

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- **Definition:** A function $f : C \rightarrow \mathbb{R}$ is strictly quasiconcave if $-f$ is strictly quasiconvex.
The Set of Minimizers

- Let $C \subset \mathbb{R}^m$ be convex, and let $f : C \rightarrow \mathbb{R}$ be strictly quasiconvex. Suppose $m$ is the minimum value attained by $f$. Prove that $\text{argmin } f := f^{-1}(m)$ is a singleton.
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Proof: We argue by *reductio ad absurdum*. 
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**Proof:** We argue by *reductio ad absurdum*. Suppose the claim is false: there are $x, y \in \text{argmin } f$ with $x \neq y$. This implies an impossible inequality:

$$m = \max\{f(x), f(y)\} > f\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq m.$$
Objectives Related to Optimization

We will study:

- Characterization of maximization of a quasiconcave function when the constraint set is convex.
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We will study:

- Characterization of maximization of a quasiconcave function when the constraint set is convex.
- The Second Welfare Theorem of Economics.
Definitions

**Definition:** Let $C \subset \mathbb{R}^m$ be convex. A function $f : C \to \mathbb{R}$ is **semistrictly quasiconcave** if it is quasiconcave and

$$f((1 - \alpha)x + \alpha y) > \max\{f(x), f(y)\}$$

whenever $x, y \in C$, $x \neq y$, $0 < \alpha < 1$, and

$$\min\{f(x), f(y)\} < \max\{f(x), f(y)\}.$$
For $\bar{x} \in C$ let:

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$$B_F(\bar{x}) := \{ x \in C : F(x) \leq F(\bar{x}) \}.$$
Convex Maximization

Theorem: Suppose that $C \subset \mathbb{R}^m$ is convex, $F : C \to \mathbb{R}$ is continuous and semistrictly quasiconcave, $G : C \to \mathbb{R}$ is quasiconvex, $\bar{x} \in C$, and $\mathcal{A}_F^o(\bar{x})$ is nonempty. Then $\bar{x}$ solves

$$\max \ F(x) \ \text{subject to} \ x \in C, \ G(x) \leq G(\bar{x})$$

if and only if there exist $p \in \mathbb{R}^m \setminus \{0\}$ and $c \in \mathbb{R}$ such that $p \cdot x \geq c$ for all $x \in \mathcal{A}_F(\bar{x})$ and $p \cdot x \leq c$ for all $x \in \mathcal{B}_G(\bar{x})$. 
Proof: We begin by noting several facts:

- \( A_F(\bar{x}) \) and \( B_G(\bar{x}) \) are convex because \( F \) is quasiconcave and \( G \) is quasiconvex.
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- $A_F^o(x)$ is the set of interior point of $A_F(x)$:
  - Since $F$ is continuous, each point of $A_F^o(x)$ is an interior point of $A_F(x)$.
  - Claim: $A_F^o(x)$ contains the interior points of $A_F(x)$:
Proof of the Claim: Let \( x \) be an interior point of \( \mathcal{A}_F(\overline{x}) \). Choose \( \hat{x} \in \mathcal{A}_F^\circ(\overline{x}) \). Then (since \( x \) is interior)

\[
x_\varepsilon := \hat{x} + (1 + \varepsilon)(x - \hat{x}) \in \mathcal{A}_F(\overline{x})
\]

for some small \( \varepsilon > 0 \). If \( F(x_\varepsilon) > F(\overline{x}) \), then
\[
F(x) \geq \min\{F(\hat{x}), F(x_\varepsilon)\} > F(\overline{x})
\]
because \( F \) is quasiconcave, and if \( F(x_\varepsilon) = F(\overline{x}) \), then
\[
F(x) > \min\{F(\hat{x}), F(x_\varepsilon)\} = F(\overline{x})
\]
because \( F \) is semistrictly quasiconcave.
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The sets $A_F(\bar{x})$ and $B_G(\bar{x})$ are convex. There is an interior point of $A_F(\bar{x})$ because $A_F^o(\bar{x})$ is nonempty. Any interior point of $A_F(\bar{x})$ is contained in $A_F^o(\bar{x})$, so there are no interior points of $A_F(\bar{x})$ in $B_G(\bar{x})$. Therefore the existence of suitable $p$ and $c$ follows from the separating hyperplane theorem.
Only If

Suppose that there exist $p \in \mathbb{R}^m \setminus \{0\}$ and $c \in \mathbb{R}$ such that $p \cdot x \geq c$ for all $x \in \mathcal{A}_F(\bar{x})$ and $p \cdot x \leq c$ for all $x \in \mathcal{B}_G(\bar{x})$. 
Only If

Suppose that there exist $p \in \mathbb{R}^m \setminus \{0\}$ and $c \in \mathbb{R}$ such that $p \cdot x \geq c$ for all $x \in A_F(x)$ and $p \cdot x \leq c$ for all $x \in B_G(x)$.

If $x \in A_F^o(x)$, then $x$ is an interior point of $A_F(x)$, so $p \cdot x > c$, which implies that $x \notin B_G(x)$. Thus $A_F^o(x) \cap B_G(x) = \emptyset$, which is the same thing as saying that $x$ solves

$$\max F(x) \text{ subject to } x \in C, G(x) \leq G(x).$$
An Exchange Economy

- Society is endowed with quantities $X_1, \ldots, X_G$ of $G$ goods which must be allocated to $C$ consumers.
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- Society is endowed with quantities $X_1, \ldots, X_G$ of $G$ goods which must be allocated to $C$ consumers.

- Denoting the quantity of good $g$ allocated to consumer $c$ by $x^c_g$, the feasible allocations are those vectors $x = (x^1, \ldots, x^C) \in (\mathbb{R}^G)^C$ satisfying $x^c_g \geq 0$ for all $c$ and $g$ and

$$x^1_g + \cdots + x^C_g \leq X_g \quad (g = 1, \ldots, G).$$
Let consumer $c$'s utility be $u^c = U^c(x^c)$. 
• Let consumer $c$’s utility be $u^c = U^c(x^c)$.

• A feasible allocation $x$ is **Pareto optimal** if there does not exist another feasible allocation $\tilde{x}$ with $U^c(\tilde{x}^c) \geq U^c(x^c)$ for all $c = 1, \ldots, C$ and $U^c(\tilde{x}^c) > U^c(x^c)$ for some $c$. 
• Let consumer $c$’s utility be $u^c = U^c(x^c)$.

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• An allocation $x$ is **supported** by a price vector $p$ if, for each $c = 1, \ldots, C$, $x_c$ solves $\max_{p \cdot x \leq I^c} U^c(x)$ where $I^c := p \cdot x^c$. 
Second Welfare Theorem

The Second Theorem of Welfare Economics:
If each $U^c$ is continuous, increasing, and quasiconcave, then any Pareto optimal allocation $x$ is supported by a price vector.
Proof

For each $c = 1, \ldots, C$ let

$$
\mathcal{B}^c := \{ y \in \mathbb{R}^G : U^c(y) \geq U^c(x^c) \}.
$$

Since $U^c$ is quasiconcave, $\mathcal{B}^c$ is convex. Let

$$
\mathcal{B} := \mathcal{B}^1 + \cdots + \mathcal{B}^C.
$$

Since sums of convex sets are convex, $\mathcal{B}$ is convex. Since each $U^c$ is increasing, $\mathcal{B}$ has an interior point.
Let

\[ A := \{ x \in \mathbb{R}^G : 0 \leq x_g \leq X_g \text{ for all } g = 1, \ldots, G \}. \]

Clearly \( A \) is convex. Since \( x \) is Pareto optimal, \( A \) contains no interior points of \( B \). Therefore, by the separating hyperplane theorem, there exists \( p \in \mathbb{R}^G \setminus \{0\} \) and \( I \in \mathbb{R} \) such that \( p \cdot x \leq I \) for all \( x \in A \) and \( p \cdot x \geq I \) for all \( x \in B \). Note that \( p \cdot \left( \sum_{c=1}^{C} x^c \right) = I \) because \( \sum_{c=1}^{C} x^c \in A \cap B \).
To produce a contradiction, suppose that, for some $c'$, $x^{c'}$ is not a solution to $\max_{p \cdot x \leq I^{c'}} U^{c'}(x)$ where $I^{c'} := p \cdot x^{c'}$. That is, there is some $\tilde{x}^{c'}$ with $p \cdot \tilde{x}^{c'} \leq I^{c'}$ and $U^{c'}(\tilde{x}^{c'}) > U^{c'}(x^{c'})$. Since $U^{c'}$ is continuous, we may perturb $\tilde{x}^{c'}$ slightly without changing this inequality, so we may assume that $p \cdot \tilde{x}^{c'} < I^{c'}$. 
Then \( \tilde{x}^{c'} + \sum_{c \neq c'} x^c \in \mathcal{B} \) and

\[
p \cdot \left( \tilde{x}^{c'} + \sum_{c \neq c'} x^c \right) = p \cdot \tilde{x}^{c'} + \sum_{c \neq c'} p \cdot x^c
\]

\[
< p \cdot x^{c'} + \sum_{c \neq c'} p \cdot x^c = \sum_{c=1}^{C} p \cdot x^c = I.
\]

This contradiction completes the proof.