Lecture 5

The Value of the Problem
Objectives

We study the optimized value of an optimization problem, aiming at:
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- Methods for computing derivatives of the value.
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• The Envelope Theorem.
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We study the optimized value of an optimization problem, aiming at:

- Methods for computing derivatives of the value.
- The Envelope Theorem.
- Useful functions in demand theory.
An Unconstrained Problem

The general form of the problem we will study is

$$\max_{x \in \mathbb{R}^n} F(x_1, \ldots, x_n, \theta_1, \ldots, \theta_m)$$

where $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$.

- Suppose that $x^*(\theta)$ solves this problem for $\theta$. This need not be differentiable, but let’s suppose it is.
- Let $V(\theta) := F(x^*(\theta), \theta)$ be the optimized value of the problem.
Applying the Chain Rule

• Apply the chain rule to compute \( \frac{dV}{d\theta}(\theta_0) \):
Applying the Chain Rule

• Apply the chain rule to compute $\frac{dV}{d\theta}(\theta_0)$:

$$\frac{dV}{d\theta}(\theta_0) = F_\theta(x^*)(\theta_0, \theta_0) + F_x(x^*)(\theta_0, \theta_0) \cdot \frac{dx^*}{d\theta}(\theta_0).$$

• What are the dimensions of these matrices of partial derivatives?
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- What are the dimensions of these matrices of partial derivatives?
- What is the first order condition for the problem?
The Envelope Theorem

- We conclude that the derivative of the value of the problem agrees with the partial derivative of the objective function:

\[ \frac{dV}{d\theta}(\theta_0) = F_\theta(x^*(\theta_0), \theta_0). \]
The Envelope Theorem

- We conclude that the derivative of the value of the problem agrees with the partial derivative of the objective function:

\[ \frac{dV}{d\theta}(\theta_0) = F_\theta(x^*(\theta_0), \theta_0). \]

- This result is the most basic form of the Envelope Theorem.
An Example: Firm Profits

- Let a firm’s output be $f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are its inputs.
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- Let a firm’s output be $f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are its inputs.
- If $p$ is the output price and $w_1, \ldots, w_n$ are the wages of the inputs, the firm’s profits are

$$\pi(p, w, x) = pf(x) - w \cdot x$$

where

$$w \cdot x = w_1 x_1 + \cdots + w_n x_n$$

is the dot product of $w$ with $x$. 
Let $x^* = x^*(p, w)$ be the profit maximizing input choice, and let

$$V(p, w) := \pi(p, w, x^*) = pf(x^*) - w \cdot x^*$$

be the maximal profit given the parameters $p$ and $w$. 
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• What is $\frac{\partial V}{\partial p}(p, w)$?
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be the maximal profit given the parameters \( p \) and \( w \).

• What is \( \frac{\partial V}{\partial p}(p, w) \)?

• For \( i = 1, \ldots, n \), what is \( \frac{\partial V}{\partial w_i}(p, w) \)?
A Story Problem

- A steel producer uses 1,000,000 tons of coal a year. If the government increases the tax on coal by $0.10/ton, and neither the price of steel nor the price of any input (including the pretax price of coal) change, approximately how much will annual profits decrease?
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• Is this an upper bound or a lower bound?
A Constrained Problem

- Now consider the constrained problem

$$\max F(x, \theta) \text{ subject to } G(x, \theta) = 0.$$
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• Let \( x^*(\theta) \) be the solution of this problem. Again, we assume differentiability.
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- Let \( x^*(\theta) \) be the solution of this problem. Again, we assume differentiability.

- Let \( V(\theta) := F(x^*(\theta), \theta) \) be the value of the problem.
Tasks

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\]

• Write the Lagrangean and the first order condition:

\[
\mathcal{L}(x, \theta) := F(x, \theta) - \lambda G(x, \theta);
\]

\[
0 = \mathcal{L}_x = F_x - \lambda G_x, \quad 0 = \mathcal{L}_\lambda = G.
\]
More Computations

• We substitute to eliminate $F_x$ from the expression for $\frac{dV}{d\theta}$:

$$\frac{dV}{d\theta} = \lambda G_x \frac{dx^*}{d\theta} + F_\theta.$$
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A General Envelope Theorem

• Substituting, we obtain \( \frac{dV}{d\theta} = -\lambda G_\theta + F_\theta = \mathcal{L}_\theta \).
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- When \( \theta \) is not an argument of \( G \), this specializes to the formula we had before:
  \( \frac{dV}{d\theta} = F_\theta \).
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• When \( \theta \) is not an argument of \( G \), this specializes to the formula we had before:
  \[ \frac{dV}{d\theta} = F_\theta. \]

• Intuitively, a change in \( \theta \) may result in some change in \( x^*(\theta) \) along the surface defined by the condition \( G(x) = 0 \), but the first order condition implies that the first order effect of such a change in zero.
Another Story Problem

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- Suppose a smoker consumes a pack of cigarettes per day. What change in annual income would have roughly the same effect on utility as a $1/pack increase in the cigarette tax?
- $1/day \times 365 \text{ days/year} = $365/\text{year}.
- Is this an upper or lower bound?
In More Detail

- Let $\theta = (p, I)$ in the consumer maximization problem $\max U(x)$ s.t. $\sum_i p_i x_i \leq I$. 
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• Let $\theta = (p, I)$ in the consumer maximization problem $\max U(x)$ s.t. $\sum_i p_i x_i \leq I$.
• Let $F(x, \theta) = U(x)$ and $G(x, \theta) = I - \sum_i p_i x_i$. 
In More Detail

- Let $\theta = (p, I)$ in the consumer maximization problem $\max U(x)$ s.t. $\sum_i p_i x_i \leq I$.
- Let $F(x, \theta) = U(x)$ and $G(x, \theta) = I - \sum_i p_i x_i$.
- If $d\theta$ is a combined change in income and the price of cigarettes that leaves the consumer no better or worse off, we have

$$0 = V_\theta d\theta = F_\theta d\theta - \lambda G_\theta d\theta = -\lambda (dI - $365/year).$$
Indirect Utility Function

- The indirect utility function $V(p, I)$ is the value of the problem

\[ \max U(x) \text{ subject to } \sum_i p_i x_i \leq I. \]
Indirect Utility Function

- The indirect utility function \( V(p, I) \) is the value of the problem

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\max U(x) \quad \text{subject to} \quad \sum_i p_i x_i \leq I.
\]

- This is an important approach to consumer theory in econometric practice.
Indirect Utility Function

- The Lagrangean is

\[ \mathcal{L}(x, p, I, \lambda) = U(x) + \lambda(I - \sum_i p_i x_i). \]
Indirect Utility Function

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\[ \mathcal{L}(x, p, I, \lambda) = U(x) + \lambda(I - \sum_i p_i x_i) \]

- Let

\[ D(p, I) = (D_1(p, I), \ldots, D_n(p, I)) \]

be the vector of demands that solve this problem.
The envelope theorem gives

\[ V_I(p, I) = \mathcal{L}_I = \lambda \quad \text{and} \quad V_{p_i} = \mathcal{L}_{p_i} = -\lambda D_i. \]
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• Therefore \( D_i(p, I) = -V_{p_i}(p, I)/V_I(p, I). \)
• The envelope theorem gives

\[ V_I(p, I) = \mathcal{L}_I = \lambda \quad \text{and} \quad V_{p_i} = \mathcal{L}_{p_i} = -\lambda D_i. \]

• Therefore \( D_i(p, I) = -V_{p_i}(p, I)/V_I(p, I). \)

• The relation between the indirect utility and the demand functions is much more direct than the relationship between the utility function and the demand functions.
Cobb-Douglas Utility

- The Cobb-Douglas utility function is
  \[ U(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \]
  where \( \alpha_1, \ldots, \alpha_n \) are positive numbers.
- We will:
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  • Compute the demand functions.
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  • Compute the demand functions.
  • Derive the indirect utility function.
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- We will:
  - Compute the demand functions.
  - Derive the indirect utility function.
  - Check the equation above.
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\[ \mathcal{L} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} + \lambda (I - \sum_{i=1}^{n} p_i x_i). \]

\[ 0 = \frac{\partial \mathcal{L}}{\partial x_i} = \alpha_i \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{x_i} - \lambda p_i \quad (i = 1, \ldots, n). \]

\[ 0 = \frac{\partial \mathcal{L}}{\partial \lambda} = I - \sum_{i=1}^{n} p_i x_i. \]
• Solve for the demands $D_1(p, I), \ldots, D_n(p, I)$. 
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\[ D_i(p, I) = \frac{\alpha_i}{\overline{\alpha} p_i} I \quad \text{where} \quad \overline{\alpha} := \sum_{j=1}^{n} \alpha_j. \]
• Solve for the indirect utility function.
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\[ V(p, I) = \kappa \frac{I^{\alpha}}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}} \text{ where } \kappa := \frac{\alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}}{\bar{\alpha}^{\bar{\alpha}}}. \]
• Solve for the indirect utility function.

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• Compute \(-V_{p_i}(p, I)/V_I(p, I)\).
The Expenditure Function

- Consider the problem

\[
\min \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad U(x) \geq u.
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The Expenditure Function

- Consider the problem
  \[ \min \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad U(x) \geq u. \]
- The vector of demands
  \[ C(p, u) = (C_1(p, u), \ldots, C_n(p, u)) \]
  is called the compensated demand function.
The Expenditure Function

- Consider the problem
  \[ \min \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad U(x) \geq u. \]

- The vector of demands
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  is called the compensated demand function.

- The value \( E(p, u) := \sum_{i=1}^{n} p_i C_i(p, u) \) of the problem is called the expenditure function.
The envelope theorem gives

\[ C_i(p, u) = \frac{\partial E}{\partial p_i}(p, u). \]
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\[ C_i(p, u) = \frac{\partial E}{\partial p_i}(p, u). \]

• Applying the chain rule, differentiate the identity \( C_i(p, u) = D_i(p, E(p, u)) \).
The envelope theorem gives

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Applying the chain rule, differentiate the identity

\[ C_i(p, u) = D_i(p, E(p, u)). \]

\[ \frac{\partial C_i}{\partial p_j} = \frac{\partial D_i}{\partial p_j} + \frac{\partial D_i}{\partial I} \frac{\partial E}{\partial p_j} = \frac{\partial D_i}{\partial p_j} + \frac{\partial D_i}{\partial I} C_j. \]
Substituting $u = V(p, I)$, noting that $C_i(p, V(p, I)) = D_i(p, I)$, and rearranging, gives the Slutsky equation:

$$\frac{\partial D_i}{\partial p_j}(p, I) = \frac{\partial C_i}{\partial p_j}(p, V(p, I)) - \frac{\partial D_i}{\partial I}(p, I)D_j(p, I).$$
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The first term is called the substitution effect.
Substituting \( u = V(p, I) \), noting that \( C_i(p, V(p, I)) = D_i(p, I) \), and rearranging, gives the Slutsky equation:

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\frac{\partial D_i}{\partial p_j}(p, I) = \frac{\partial C_i}{\partial p_j}(p, V(p, I)) - \frac{\partial D_i}{\partial I}(p, I)D_j(p, I).
\]

The first term is called the substitution effect.

The second term is called the income effect.
The Cobb-Douglas Case

- Find the expenditure function by solving the identity

\[ I = E(p, V(p, I)) = E(p, \kappa \frac{I^\alpha}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}). \]
The Cobb-Douglas Case

- Find the expenditure function by solving the identity

\[ I = E(p, V(p, I)) = E(p, \kappa \frac{I^\alpha}{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}). \]

\[ E(p, u) = \left( \frac{u \cdot p_1^{\alpha_1} \cdots p_n^{\alpha_n}}{\kappa} \right)^{1/\alpha}. \]