Economics 4113
Intro to Math Econ

Lecture 1

Two Good Consumer Maximization
Objectives

For two good consumer optimization we study:
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- Practical solution methods.
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- Practical solution methods.
- Preliminary understanding of marginal analysis.
- Preliminary acquaintance with the relation between concavity and optimization.
- Examples illustrating local vs. global analysis.
The Prototypical Example

The simplest and most standard example is

$$\max U(x_1, x_2) := x_1 x_2$$

subject to  $$x_1 + x_2 \leq 2, \ x_1, x_2 \geq 0.$$
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- A function $f : \mathbb{R}^2_\geq \to \mathbb{R}$ is **monotonic increasing** if $f(y_1, y_2) \geq f(x_1, x_2)$ whenever $y_1 \geq x_1$ and $y_2 \geq x_2$. 
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• Substitute to reduce to a one variable problem.
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- Substitute to reduce to a one variable problem.
- Solve.
A Parameterized Example

Let $p_1$, $p_2$, and $I$ be positive numbers. We now consider a problem for which these are parameters:

$$\max U(x_1, x_2) := \ln x_1 + 2 \ln x_2$$

subject to $p_1 x_1 + p_2 x_2 \leq I$, $x_1, x_2 \geq 0$. 
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• Solve.
The Arbitrage Argument

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- To be precise, an arbitrage is a portfolio of financial assets that has zero cost (because it combines long and short positions) and a positive payoff with probability one.
- For example, simultaneously buying and selling the same stock at different prices on different stock exchanges.
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• We will look at the original problem from the point of view of deviations from the optimum.
We will use the first example, with general prices:

$$\max U(x_1, x_2) := x_1 x_2$$

subject to   $$p_1 x_1 + p_2 x_2 \leq 2, \ x_1, x_2 \geq 0.$$
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- The optimal bundle is \((1/p_1, 1/p_2)\), but we will not solve the problem as we did above.
- Instead, we will use “arbitrage” style analysis to demonstrate that this point solves the problem.
• Express the utility as a function of $\Delta x_1$ and $\Delta x_2$, where

$$\Delta x_1 := x_1 - 1/p_1 \quad \text{and} \quad \Delta x_2 := x_2 - 1/p_2.$$
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• In checking whether $(1/p_1, 1/p_2)$ is an optimum, does it make sense to look only at pairs $(\Delta x_1, \Delta x_2)$ with $p_2 \Delta x_2 = -p_1 \Delta x_1$?
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• What happens when we set $\Delta x_2 = -p_1 \Delta x_1 / p_2$ in the utility function?
Marginal Analysis

Approximately (when $\Delta x_1$ and $\Delta x_2$ are small) the change in utility resulting from switching from $(1, 1)$ to $(1 + \Delta x_1, 1 + \Delta x_2)$ is

$$MU_1(x_1, x_2) \cdot \Delta x_1 + MU_2(x_1, x_2) \cdot \Delta x_2.$$

• Here

$$MU_i(x_1, x_2) := \frac{\partial U}{\partial x_i}(x_1, x_2)$$

is the marginal utility of additional consumption of good $i$. 
Partial Derivatives

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. What is the partial derivative of $f$ with respect to $x_i$ at $x$?
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• Recall that the partial derivative of $f$ with respect to $x_i$ at $x$ is

$$\frac{\partial f}{\partial x_i}(x) := \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x)}{h}.$$
Cautions

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Cautions

- **Warning:** The partial derivatives at $x$ may not be defined.

- **Warning:** Even if all $n$ partial derivatives of $f$ exist at every $x \in \mathbb{R}^n$, it can still happen that $f$ is not “well behaved.”

- **Warning:** Partial derivatives will be very important in this course. This would be a good time to refresh your understanding.
What relation must hold between $MU_1(x_1, x_2)$, $MU_2(x_1, x_2)$, $p_1$, and $p_2$ if

$$MU_1(x_1, x_2) \cdot \Delta x_1 + MU_2(x_1, x_2) \cdot \Delta x_2 \leq 0$$

for all $(\Delta x_1, \Delta x_2)$ satisfying $p_1 \Delta x_1 + p_2 \Delta x_2 \leq 0$?
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Prices must be proportional to marginal utilities:

$$\frac{MU_1(x_1, x_2)}{MU_2(x_1, x_2)} = \frac{p_1}{p_2}.$$
Another Example

The next example is simple, but a bit tricky:

\[
\max U(x_1, x_2) := x_1^2 + x_2^2
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subject to \(x_1 + x_2 \leq 2, \ x_1, x_2 \geq 0\).
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• Is \( U \) monotonic increasing? Substitute to reduce to one variable and solve.

• A *corner solution* is a solution of the problem in which one good is not consumed at all.
Yet Another Example

The next example is not so simple and quite tricky:

\[
\max U(x_1, x_2) := x_1^2 + x_2^2 + 3(x_1 x_2)^{3/2}
\]

subject to \( x_1 + x_2 \leq 1, \ x_1, x_2 \geq 0. \)
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Local vs. Global

- A point $x$ in the budget set is a local maximum if, for some $\varepsilon > 0$, $U(x) > U(x')$ for all $x'$ in the intersection of the budget set and the $\varepsilon$-ball around $x$. 
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- A point $x$ in the budget set is a *local maximum* if, for some $\varepsilon > 0$, $U(x) > U(x')$ for all $x'$ in the intersection of the budget set and the $\varepsilon$-ball around $x$.

- Unless there is some additional information, to solve a problem with multiple local maxima one must find them all and compare the value of the utility function at each.
Concavity

- $U$ is concave if

$$U((1 - \alpha)x + \alpha x') \geq (1 - \alpha)U(x) + \alpha U(x')$$

for all $x$ and $x'$ in the domain of $U$ and all $\alpha \in [0, 1]$. 
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- If $U$ is concave, can the problem “$\max U(x)$ subject to $p_1x_1 + p_2x_2 \leq I, x_1, x_2 \geq 0$” have multiple local maxima at which $U$ takes on different values?
- \( U \) is strictly concave if

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U((1 - \alpha)x + \alpha x') > (1 - \alpha)U(x) + \alpha U(x')
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for all distinct \( x \) and \( x' \) in the domain of \( U \) and all \( \alpha \in (0, 1) \).
• *U* is strictly concave if

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for all distinct *x* and *x'* in the domain of *U* and all \( \alpha \in (0, 1) \).

• If *U* is strictly concave, can the problem “\( \max U(x) \) subject to \( p_1 x_1 + p_2 x_2 \leq I \), \( x_1, x_2 \geq 0 \)” have multiple local maxima?