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In economics it is common to maximize a concave function on a convex set.

- The separating hyperplane theorem gives conditions for maximization that are both necessary and sufficient.
- These methods can also be used to study how the value of the problem changes as the constraints are varied.
• We also study maximization of quasiconcave functions on convex sets.
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• These ideas will be applied to linear programming.
Superdifferentials

Let $C \subset \mathbb{R}^m$ be convex.
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Superdifferentials

Let $C \subset \mathbb{R}^m$ be convex.

- We will always assume that $C$ has an interior point.

Let $F : C \to \mathbb{R}$ be concave.

- We will always assume that $F$ is continuous. (If $C$ is open, all concave functions on $C$ are continuous, but we won’t prove this.)
Let \( D \) be the set defined as
\[
D := \{ (x, v) \in C \times \mathbb{R} : v \leq F(x) \}.
\] Since \( F \) is concave, \( D \) is convex. If \( x \) is an interior point of \( C \), then, since \( F \) is continuous, \((x, v)\) is an interior point of \( D \) whenever \( v < F(x) \).
Let

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- Since \( F \) is concave, \( \mathcal{D} \) is convex.
- If \( x \) is an interior point of \( C \), then, since \( F \) is continuous, \( (x, v) \) is an interior point of \( \mathcal{D} \) whenever \( v < F(x) \). Explain.
Fixing $x_0 \in C$, let

$$\mathcal{E} := \{(x_0, F(x_0))\}.$$
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- Since it has only one element, $\mathcal{E}$ is convex.
- For any $x \in C$, $(x_0, F(x_0))$ is not an interior point of $\mathcal{D}$ because $(x_0, F(x_0) + \varepsilon) \notin \mathcal{D}$ for arbitrarily small $\varepsilon > 0$. 
The hypotheses of the separating hyperplane theorem hold: $\mathcal{D}$ and $\mathcal{E}$ are convex, $\mathcal{D}$ has an interior point, and no point of $\mathcal{E}$ is an interior point of $\mathcal{D}$. Therefore there exist $a_2 \in \mathbb{R}^{m+1}$ and $b_2 \in \mathbb{R}$ such that $a_2^\top(x_0; f(x_0)) \geq b_2 \geq a_2^\top(x; v)$ for all $(x; v) \in \mathcal{D}$. (Since $(x_0; f(x_0)) \in \mathcal{D}$, necessarily $b_2 = a_2^\top(x_0; f(x_0))$.)
The hypotheses of the separating hyperplane theorem hold: $\mathcal{D}$ and $\mathcal{E}$ are convex, $\mathcal{D}$ has an interior point, and no point of $\mathcal{E}$ is an interior point of $\mathcal{D}$.

Therefore there exist $a \in \mathbb{R}^{m+1} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$a \cdot (x_0, F(x_0)) \geq b \geq a \cdot (x, v)$$

for all $(x, v) \in \mathcal{D}$. (Since $(x_0, F(x_0)) \in \mathcal{D}$, necessarily $b = a \cdot (x_0, F(x_0))$.)
We write $a := (-\alpha, \beta)$ where $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$. Then

$$-\alpha \cdot x_0 + \beta F(x_0) \geq -\alpha \cdot x + \beta v$$

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• We write $a := (-\alpha, \beta)$ where $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$. Then

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for all $(x, v) \in D$.

• Clearly $\beta \geq 0$. If $x_0$ is an interior point of $C$, then necessarily $\beta > 0$. Why?
• We write $a := (-\alpha, \beta)$ where $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$. Then

$$-\alpha \cdot x_0 + \beta F(x_0) \geq -\alpha \cdot x + \beta v$$

for all $(x, v) \in \mathcal{D}$.

• Clearly $\beta \geq 0$. If $x_0$ is an interior point of $C$, then necessarily $\beta > 0$. Why?

• Since we can replace $\alpha$ and $\beta$ with $\alpha/\beta$ and 1, there is no loss of generality in assuming that $\beta = 1$. 
Thus we have shown the existence of an \( \alpha \in \mathbb{R}^m \) such that \( F(x_0) + \alpha \cdot (x - x_0) \geq v \) for all \( (x, v) \in D \), i.e., \( F(x_0) + \alpha \cdot (x - x_0) \geq F(x) \) for all \( x \in C \). Such an \( \alpha \) is called a supergradient of \( F \) at \( x_0 \).
• Thus we have shown the existence of an \( \alpha \in \mathbb{R}^m \) such that \( F(x_0) + \alpha \cdot (x - x_0) \geq v \) for all \((x, v) \in D\), i.e., \( F(x_0) + \alpha \cdot (x - x_0) \geq F(x) \) for all \( x \in C \). Such an \( \alpha \) is called a supergradient of \( F \) at \( x_0 \).

• If \( F \) is differentiable at \( x_0 \), then

\[
\alpha = \left( \frac{\partial F}{\partial x_1}(x_0), \ldots, \frac{\partial F}{\partial x_m}(x_0) \right).
\]

Explain why in detail.
Maximization

- Clearly $x_0$ is a solution of $\max_{x \in C} F(x)$ if and only if $0 \in \mathbb{R}^m$ is a supergradient of $F$. This does not depend on $F$ being differentiable.
Maximization

• Clearly \( x_0 \) is a solution of \( \max_{x \in C} F(x) \) if and only if \( 0 \in \mathbb{R}^m \) is a supergradient of \( F \). This does not depend on \( F \) being differentiable.

• If, in addition, \( F \) is differentiable at \( x_0 \), then “\( x_0 \) solves \( \max_{x \in C} F(x) \)” implies the first order condition

\[
\frac{\partial F}{\partial x_1}(x_0) = \cdots = \frac{\partial F}{\partial x_m}(x_0) = 0.
\]
Constrained Maximization

- We continue to assume that $C$ is convex and has an interior point, and that $F: C \rightarrow \mathbb{R}$ is concave and continuous.
Constrained Maximization

- We continue to assume that $C$ is convex and has an interior point, and that $F : C \rightarrow \mathbb{R}$ is concave and continuous.
- In addition, suppose that $G_1, \ldots, G_k : C \rightarrow \mathbb{R}$ are convex.
Example

Suppose that \((K, L)\) is a firm’s vector of inputs.

- \(C = \mathbb{R}^2_+\).
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- \(C = \mathbb{R}^2_+\).
- \(F(K, L) = pf(K, L) - rK - wL\) if the firm’s profit, where \(f : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+\) is a concave production function, and \((r, w) \in \mathbb{R}^2_+\) is the vector of factor prices.
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- \(G(K, L) = g(K, L) - \bar{c} \in \mathbb{R}\) is the amount of a particular pollutant, in excess of an allowed quantity \(\bar{c}\), emitted by the firm.
Let

\[ A := \{ x \in C : G_1(x) \leq 0, \ldots, G_k(x) \leq 0 \} \].

Why is \( A \) convex?
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Why is \( A \) convex?

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**Why is \( A \) convex?**

• Suppose that \( x_0 \) solves \( \max_{x \in A} F(x) \).

• Let

\[ B := \{ x \in C : F(x) \geq F(x_0) \} \].
• Let

\[ A := \{ x \in C : G_1(x) \leq 0, \ldots, G_k(x) \leq 0 \} . \]

**Why is** \( A \) **convex?**

• Suppose that \( x_0 \) solves \( \max_{x \in A} F(x) \).

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\[ B := \{ x \in C : F(x) \geq F(x_0) \} . \]

• Since \( F \) is concave, \( B \) is convex.
Suppose there is some $x \in C$ with $F(x) > F(x_0)$. Then $B$ has an interior point. Why? No point of $A$ is an interior point of $B$. Why? The separating hyperplane theorem implies the existence of $\hat{r} \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $\hat{r} \cdot x \leq b$ for all $x \in A$ and $\hat{r} \cdot x \geq b$ for all $x \in B$. 

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- Then $B$ has an interior point. Why?
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- Then $B$ has an interior point. Why?
- No point of $A$ is an interior point of $B$. Why?
- The separating hyperplane theorem implies the existence of $\alpha \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $\alpha \cdot x \leq b$ for all $x \in A$ and $\alpha \cdot x \geq b$ for all $x \in B$. 
Example: Analysis

Let \( f(K, L) := \ln K + \ln L \), let \( g(K, L) := 2K + L \), and let \( \bar{c} := 16 \).

- **Draw \( A \).**
Example: Analysis

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- **Draw** \( A \).
- **Find the solution of** \( \max_{K, L \geq 0} F(K, L) \). **When is it in** \( A \)?
Example: Analysis

Let $f(K, L) := \ln K + \ln L$, let $g(K, L) := 2K + L$, and let $\bar{c} := 16$.

- **Draw $A$.**
- **Find the solution of** $\max_{K,L\geq 0} F(K, L)$. **When is it in $A$?**
- **Solve the constrained problem using the Lagrangean, and sketch the separating hyperplane.**
We now study the parameterized problem

\[
\max \ G_1(x) \cdot c_1, \ldots, G_k(x) \cdot c_k \ F(x).
\]

Let \( X(c) \) be the solution of the problem.

Let \( V(c) = F(X(c)) \) be the value of the problem.

Solve for these in the example.
Concave Programming

We now study the parameterized problem

$$\max \ F(x).$$

$$G_1(x) \leq c_1, \ldots, G_k(x) \leq c_k$$

- Let $X(c)$ be the solution of the problem.
Concave Programming

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$$\max_{G_1(x) \leq c_1, \ldots, G_k(x) \leq c_k} F(x).$$

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be the value of the problem.
- Solve for these in the example.
$V$ is Concave and Increasing

- Consider $c_0, c_1$ and $\alpha \in [0, 1]$. Let

  \[ x_0 := X(c_0) \quad \text{and} \quad x_1 := X(c_1). \]
$V$ is Concave and Increasing

- Consider $c_0, c_1$ and $\alpha \in [0, 1]$. Let

$$x_0 := X(c_0) \quad \text{and} \quad x_1 := X(c_1).$$

- Idea: $V((1 - \alpha)c_0 + \alpha c_1)$ is bounded below by the possibility of choosing $(1 - \alpha)x_0 + \alpha x_1$. 
\( V \) is Concave and Increasing

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- Idea: \( V((1 - \alpha)c_0 + \alpha c_1) \) is bounded below by the possibility of choosing \((1 - \alpha)x_0 + \alpha x_1\).

- This is feasible: for \( i = 1, \ldots, k \) we have

\[
G_i((1 - \alpha)x_0 + \alpha x_1) \leq (1 - \alpha)G_i(x_0) + \alpha G_i(x_1)
\leq (1 - \alpha)c_0 + \alpha c_1.
\]
Therefore

\[ V((1 - \alpha)c_0 + \alpha c_1) \geq F((1 - \alpha)x_0 + \alpha x_1) \]

\[ \geq (1 - \alpha)F(x_0) + \alpha F(x_1) \]

\[ = (1 - \alpha)V(c_0) + \alpha V(c_1). \]
Therefore

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\[ \geq (1 - \alpha)F(x_0) + \alpha F(x_1) \]
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To see that \( V \) is increasing, suppose that \( c \leq c' \).
• Therefore

\[ V((1 - \alpha)c_0 + \alpha c_1) \geq F((1 - \alpha)x_0 + \alpha x_1) \geq (1 - \alpha)F(x_0) + \alpha F(x_1) = (1 - \alpha)V(c_0) + \alpha V(c_1). \]

• To see that \( V \) is increasing, suppose that \( c \leq c' \). Then \( G(X(c)) \leq c \leq c' \), so

\[ V(c') \geq F(X(c)) = V(c). \]
A Separating Hyperplane

Fix a particular $c^*$ and $v^* := V(c^*)$. We separate the following two sets:

- $A := \{ (c, v) \in \mathbb{R}^k \times \mathbb{R} : v \leq V(c) \}$;
A Separating Hyperplane

Fix a particular $c^*$ and $v^* := V(c^*)$. We separate the following two sets:

- $\mathcal{A} := \{ (c, v) \in \mathbb{R}^k \times \mathbb{R} : v \leq V(c) \}$;
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- $\mathcal{A}$ and $\mathcal{B}$ are clearly convex, and the interior of $\mathcal{B}$ is nonempty and contains no points of $\mathcal{A}$.
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$\mathcal{A}$ and $\mathcal{B}$ are clearly convex, and the interior of $\mathcal{B}$ is nonempty and contains no points of $\mathcal{A}$.

- **Draw these sets for the example.**
The separating hyperplane theorem implies that there is

\[(\lambda, \nu) \in \mathbb{R}^k \times \mathbb{R} \setminus \{(0, 0)\}\]

such that

\[\nu v - \lambda c \leq \nu v^* - \lambda c^* \leq \nu v' - \lambda c'\]

for all \((c, v) \in \mathcal{A}\) and all \((c', v') \in \mathcal{B}\).
The separating hyperplane theorem implies that there is
\[(\lambda, \iota) \in \mathbb{R}^k \times \mathbb{R} \setminus \{(0, 0)\}\]
such that
\[\iota \nu - \lambda c \leq \iota \nu^* - \lambda c^* \leq \iota \nu' - \lambda c'\]
for all \((c, \nu) \in \mathcal{A}\) and all \((c', \nu') \in \mathcal{B}\).

- Given the definition of \(\mathcal{B}\), we must have \(\lambda \geq 0\) (that is, \(\lambda_i \geq 0\) for all \(i\)) and \(\iota \geq 0\).
The Slater Condition

- It is possible that \( \ell = 0 \), but this case is usually not interesting.
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The Slater Condition

• It is possible that $\iota = 0$, but this case is usually not interesting.

• The Slater condition is that there is some $x_0 \in C$ with $G(x_0) \ll c^*$.

• Why does this imply that $\iota > 0$?

• Since $F(x_0) \leq V(G(x_0))$, we have

$$ (G(x_0), F(x_0)) \in A, $$

so

$$ \iota F(x_0) - \lambda G(x_0) \leq \iota v^* - \lambda c^*. $$

But if $\iota = 0$, then $\lambda_i > 0$ for at least one $i$, implying that

$$ \lambda(c^* - G(x_0)) > 0. $$
A Remark on Terminology

Dixit’s use of the term “indivisibility” in connection with the possibility that $\varepsilon = 0$ is strange. A good is divisible (sometimes “infinitely divisible”) if arbitrary small quantities make sense. Indivisible goods are the ones that come in integer quantities.
Shadow Prices

- If $(\lambda, \nu)$ defines a separating hyperplane, then so does $\alpha(\lambda, \nu)$ for any $\alpha > 0$. 

Therefore (given the Slater condition, or simply that $\nu > 0$), we may normalize by assuming that $\nu = 1$.

For each possible $c$ we have $(c; V(c)) \in A$ and thus $V(c) - \lambda c \geq V(c - \delta)$.

That is, $X(c - \delta)$ solves the unconstrained problem $\max F(x) - \lambda G(x)$. 
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Shadow Prices

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- For each possible \(c\) we have \((c, V(c)) \in A\) and thus \(V(c) - \lambda c \leq V(c^*) - \lambda c^*\).
- That is, \(X(c^*)\) solves the unconstrained problem \(\max F(x) - \lambda G(x)\).
Tradable Pollution Permits

Decentralization is an important theme of economics. It has two main components:

1. Achieve social goals by providing incentives (e.g., Pigouvian taxes) rather than detailed directives.
2. Use markets to compute incentives. (Shadow prices become actual prices.)

In pollution permit markets there have been low prices and government buy backs.
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Sufficient Conditions

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Sufficient Conditions

Sufficient conditions for $\bar{x}$ to solve $\max_{G(x) \leq c} F(x)$ are that there exists $\lambda \geq 0$ such that

- $\bar{x}$ solves $\max_{x \in C} F(x) - \lambda G(x)$;
- $G(\bar{x}) \leq c$ with complementary slackness: $\lambda(c - G(\bar{x})) = 0$. 

Proof:

If $G(\bar{x}) \cdot c$ then $\lambda G(\bar{x}) = \lambda c$ and $\lambda (c - G(\bar{x})) = 0$, so $F(\bar{x}) = F(x) + \lambda G(\bar{x})$: 

Sufficient Conditions

Sufficient conditions for $\bar{x}$ to solve $\max_{G(x) \leq c} F(x)$ are that there exists $\lambda \geq 0$ such that

- $\bar{x}$ solves $\max_{x \in \mathcal{C}} F(x) - \lambda G(x)$;
- $G(\bar{x}) \leq c$ with complementary slackness: $\lambda(c - G(\bar{x})) = 0$.

**Proof:** If $G(x) \leq c$ then $\lambda G(\bar{x}) = \lambda c$ and $\lambda(c - G(x)) \geq 0$, so

$$F(\bar{x}) \geq F(x) + \lambda(G(\bar{x}) - G(x)) = F(x).$$
Linear Programming

A standard form of linear programming is the problem

$$\max a^T x \quad \text{subject to} \quad Bx \leq c \quad \text{and} \quad x \geq 0.$$
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Linear Programming

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- It is theoretically important for many reasons, e.g., the linear approximation of the neighborhood of a solution of a nonlinear problem is a linear program.
- Linear programming has many industrial applications.
Input-Output Interpretation

If the components of $a$, $B$, and $c$ are all nonnegative, then these objects can be interpreted as an input-output economy:

\[
\begin{align*}
\text{a} &= (a_1; \ldots; a_m) \\
\text{B} &= \text{an} \times m \text{input requirement matrix with } b_{ij} \text{ the amount of input } i \text{ consumed by the production of one unit of output } j \\
\text{c} &= (c_1; \ldots; c_n)
\end{align*}
\]
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- $c = (c_1, \ldots, c_n)$ is a vector of input supplies.
Feasibility and Optimality

- A feasible solution is a point $x \in \mathbb{R}^m$ satisfying $Bx \leq c$ and $x \geq 0$. 

The problem is:

- infeasible if there are no feasible solutions;
- unbounded if there are feasible $x$ with $a^T x$ arbitrarily large.

Task: give an example of each sort of failure.

Fact: if the problem is feasible and bounded, a solution exists.
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Feasibility and Optimality

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- The problem is:
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- Task: give an example of each sort of failure.
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- The problem is:
  - **infeasible** if there are no feasible solutions;
  - **unbounded** if there are feasible \( x \) with \( a^T x \) arbitrarily large.
- **Task**: give an example of each sort of failure.
- **Fact**: if the problem is feasible and bounded, a solution exists.
Lagrangean Analysis

- The Lagrangean is $\mathcal{L} := a^T x + \lambda^T (c - B x)$. 
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The Dual Problem

The dual problem is

$$\max -c^T y \quad \text{subject to} \quad -B^T y \leq -a \text{ and } y \geq 0.$$
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- In the input-output context this expresses: competition to supply inputs (in multiples of \( c \)) drives the price of \( c \) to the minimum consistent with bounded demand.
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- In the input-output context this expresses: competition to supply inputs (in multiples of $c$) drives the price of $c$ to the minimum consistent with bounded demand.

- Task: for the infeasible and unbounded programs found above, determine the status of the dual problems.
The Dual of the Dual

The transformation passing from primal to dual is

\[(m, n, a, B, c) \rightarrow (n, m, -c, -B^T, -a).\]
The Dual of the Dual

The transformation passing from primal to dual is

$$(m, n, a, B, c) \rightarrow (n, m, -c, -B^T, -a).$$

- Applying this transformation twice returns us to the original problem: the dual of the dual is the primal.
A Key Computation

Suppose that $x$ is a feasible for the primal problem and $y$ is feasible for the dual problem:

$$
x \geq 0; \quad y \geq 0; \quad c - Bx \geq 0; \quad B^T y - a \geq 0.
$$

This gives $a^T x \leq y^T Bx \leq y^T c$.

Thus both problems are bounded.
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Optimality

- Note that $a^T x = y^T B x = y^T c$ if and only if the complementary slackness conditions hold:

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- In this case $x$ must be optimal, since for all feasible $\tilde{x}$ we have $a^T \tilde{x} \leq y^T c$. Symmetrically, $a^T x \leq \tilde{y}^T c$ for all feasible $\tilde{y}$, so $y$ is optimal.
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The Remaining Question

- We now know that both problems are feasible and bounded if either: (a) both problems are feasible; (b) one problem is feasible and bounded.
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- From this it follows that if one problem is unbounded, the other must be infeasible.
- Can both problems be infeasible?
- Maybe or maybe not, but if not, then for sure the SHT will say why.
Simultaneous Infeasibility

- Let

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a = \begin{bmatrix} 5 \\ 6 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}; \quad c = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.
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- There is no \( x \geq 0 \) with \( Bx \leq c \).
- There is no \( y \geq 0 \) with \( -B^T y \leq -a \).
- Both problems can be infeasible!
Summary

The following are equivalent:

The primal is feasible and bounded.

Both problems are feasible.

The dual is feasible and bounded.

If $x$ is feasible for the primal and $y$ is feasible for the dual, then $a^T x \cdot y^T c$, with equality if and only if the dual slackness conditions $0 = (B^T y - a)^T x$ and $0 = y^T (c - B x)$ both hold.
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