Economics 3012
Strategic Behavior
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Lecture 8

Topics

• Problem Set 7

• Two Problems on Mixed Equilibrium

• Sequential Games with Simultaneous Moves
  – Exiting a Declining Industry
  – Voting by Binary Agenda
  – Chance Moves
  – A Sequential Duel
Problem 1: Do Exercise 187.1 of Osborne.

Problem Statement: The set of possible policies is the set of real numbers. The game proceeds as follows:

- A committee makes a proposal.
- The legislature either:
  - accepts, in which case the proposal is implemented, or
  - rejects, in which case the outcome is the status quo $y_0$.

The committee and the legislature are each assumed to have single peaked preferences.
- The legislature’s favorite outcome is 0.
- The committee’s favorite outcome is $y_c > 0$. 
Analysis:

- In a subgame perfect equilibrium the legislature will accept any proposal that it strictly prefers to the status quo and reject any proposal that is strictly worse than the status quo from its point of view.

- Therefore the committee will propose its favorite proposal from those that the legislature is willing to accept.

- If $y_0 < 0$, then an increase in $y_0$ improves it in the eyes of the legislature, thereby diminishing the set of policies that the legislature prefers to the status quo, with the possible consequence that the committee’s favorite such policy is further from $y_c$ and closer to 0.
Problem 2: Do Exercise 189.1 of Osborne.

Problem Statement: Firm i’s cost is $q_i^2$, and the price is $P(q_1 + q_2)$ where

$$P(Q) = \begin{cases} 
\alpha - Q, & Q \leq \alpha, \\
0, & Q > \alpha.
\end{cases}$$

Find the subgame perfect equilibrium of the Stackelberg duopoly game and compare it to the Nash equilibrium of the Cournot game.

Analysis: We find the subgame perfect equilibrium using backwards induction:

- For given $q_1$, we solve for Firm 2’s best response by setting
  $$0 = \frac{d[q_2(\alpha-(q_1+q_2))-q_2^2]}{dq_2} = \alpha - q_1 - 4q_2,$$
  finding that $b_2(q_1) = \frac{\alpha - q_1}{4}$. 
• To find Firm 1’s optimal quantity in the Stackelberg game we set

\[
0 = \frac{d[q_1(\alpha-(q_1+(\alpha-q_1)/4))-q_1^2]}{dq_1} = \frac{3}{4} \alpha - \frac{7}{2} q_1.
\]

– This gives \( q_1 = \frac{3}{14} \alpha \) and consequently \( q_2 = \frac{11}{56} \alpha \).

– Total output is \( \frac{23}{56} \alpha \).

To find the Cournot output we solve the system of equations

\[
q_2 = \frac{\alpha-q_1}{4} \quad \text{and} \quad q_1 = \frac{\alpha-q_2}{4}.
\]

• The solution is \( q_1 = q_2 = \frac{1}{5} \alpha \).

• We have \( \frac{11}{56} \alpha < \frac{1}{5} \alpha < \frac{12}{56} \alpha \) and \( \frac{23}{56} > \frac{2}{5} \alpha \), so the output in the Stackelberg case is (slightly) greater than the output in the Cournot case.
Problem 3: Do Exercise 201.2 of Osborne.

Problem Statement: Two players take turns removing stones from a pile.

• A player may take either one or two stones at her turn.

• The winner is the one to take the last stone.

Analysis: Let $s(n) \in \{L, W\}$ be the status of the game with $n$ stones remaining:

• The status is $W$ if the first player to move can force a win.

• Otherwise the status is $L$.

• Then $s(1) = s(2) = W$. When $n = 3$, the player to move must leave either one or two, so $s(3) = L$. When $n$ is either 4 or 5, the player to move can leave 3, so $s(4) = s(5) = W$. And so forth...
• In general, for $n \geq 1$ we have

$$s(n) = \begin{cases} 
L, & s(n - 2) = s(n - 1) = W, \\
W, & \text{otherwise.} 
\end{cases}$$

• Using mathematical induction one can show that

$$s(n) = \begin{cases} 
L, & n \text{ is divisible by } 3, \\
W, & \text{otherwise.} 
\end{cases}$$
Problem 4: Do Exercise 202.1 of Osborne.

Problem Statement: There are $n$ lions.

- The first can eat a piece of prey, but then becomes vulnerable to being eaten by the second.
- In turn, if lion $i$ eats lion $i - 1$, it becomes vulnerable to being eaten by lion $i + 1$.

Each lion prefers eating to going hungry, but going hungry is better than being eaten.

Analysis: A single lion simply eats the prey. If there are two lions, the first won’t eat. But if there are three lions the first can eat the prey because the second won’t eat it for fear of being eaten by the third. In general, as can be verified by induction, the subgame perfect equilibrium if for lion $i$ to eat if $n - i$ is even and to not eat if $n - i$ is odd.
Exercise 141.3 of Osborne

Problem Statement: Party A and Party B each have to decide which of three districts to devote campaign resources to.

- If Party A contest district \( i \) and Party B contests another district, Party A gains \( a_i \) votes, where

\[
a_1 > a_2 > a_3.
\]

- If Party A and Party B contest the same district, Party A wins no votes.

Analysis: The game has the following normal form:

\[
\begin{pmatrix}
d_1 & D_1 & D_2 & D_3 \\
d_2 & (0, 0) & (a_1, -a_1) & (a_1, -a_1) \\
d_3 & (a_2, -a_2) & (0, 0) & (a_2, -a_2)
\end{pmatrix}
\]

\[
\begin{pmatrix}
d_3 & (a_3, -a_3) & (a_3, -a_3) & (0, 0)
\end{pmatrix}
\]
• There are no pure strategy equilibria:
  – Party B’s best response to any pure strategy of Party A is to contest the same district.
  – Party A’s best response to any pure strategy of Party B is to contest the more valuable of the two other districts.

We now look for totally mixed equilibria.

• Suppose that there is a totally mixed equilibrium in which Party B is mixing with probabilities $q_1$, $q_2$, and $q_3$.

• In order Party A to be indifferent between all three mixed strategies it must be the case that

$$
(q_2 + q_3)a_1 = (q_1 + q_3)a_2 = (q_1 + q_2)a_3.
$$

• Since $q_1 + q_2 + q_3 = 1$, this can be rewritten as

$$
(1 - q_1)a_1 = (1 - q_2)a_2 = (1 - q_3)a_3 = e
$$

where $e$ is the expected utility.
• Then \((1 - q_i) = e/a_i\) for \(i = 1, 2, 3\), and summing over \(i\) gives

\[
\frac{e}{a_1} + \frac{e}{a_1} + \frac{e}{a_1} = 3 - (q_1 + q_2 + q_3) = 2.
\]

• Therefore

\[
e = \frac{2}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}.
\]

• Therefore

\[
q_i = 1 - \frac{e}{a_i} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - \frac{2}{a_i}}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}.
\]

• Letting \(p_1, p_2,\) and \(p_3\) be Party A’s mixing probabilities, essentially the same calculation gives

\[
p_i = 1 - \frac{e}{a_i} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - \frac{2}{a_i}}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}.
\]

• This will make sense if

\[
\frac{1}{a_1} + \frac{1}{a_2} > \frac{1}{a_3}.
\]
The other possibility is that the two parties mix between districts 1 and 2.

- In this case we have

\[ q_2 a_1 = q_1 a_2. \]

- Again let \( e \) be the expected utility. Then

\[
1 = q_1 + q_2 = e \left( \frac{1}{a_1} + \frac{1}{a_2} \right),
\]

so

\[
q_1 = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2}} \quad \text{and} \quad q_2 = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2}}.
\]

- This will make sense if

\[
a_3 \leq e = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2}}.
\]
Exercise 142.1 of Osborne

Problem Statement: The game has the following normal form:

\[
\begin{array}{ccc}
A & B \\
A & (1,1,1) & (0,0,0) \\
B & (0,0,0) & (0,0,0) \\
A & (0,0,0) & (0,0,0) \\
B & (4,4,4) & (4,4,4) \\
\end{array}
\]

Analysis:

- There are two pure strategy equilibria \((A, A, A)\) and \((B, B, B)\).
- In any other equilibrium all agents must be playing mixed strategies.
• Let \( p, q, \) and \( r \) be the probabilities that the three agents assign to the first pure strategy.

• Then:

\[
qr = 4(1-q)(1-r),
\]
\[
pr = 4(1-p)(1-r),
\]
\[
pq = 4(1-p)(1-q).
\]

– Solving the second of these for \( p \) and the third for \( q \) shows that \( p = q \). Similarly, \( p = r \), and \( q = r \).

• We now have

\[
p^2 = 4(1-p)^2.
\]

– Clearly this is solved by setting \( p = 2/3 \).

– Since \( p^2 \) is increasing in \( p \) and \( 4(1-p)^2 \) is decreasing, there is no other solution between 0 and 1.
The models and examples in Chapter 7 involve a more general notion of extensive game in which there are points where several players move simultaneously.

The game begins with player 1 choosing between reading a book and going to a concert.

- If player 1 reads a book the game ends.
- If player 1 decides to go to a concert, then the two players simultaneously each decide whether to go to hear Bach or Stravinsky.
Description in Terms of Histories

In this example:

- The set of players is \( \{1, 2\} \).
- The set of terminal histories is
  \[
  \{ \text{Book}, (\text{Concert}, (B, B)), (\text{Concert}, (B, S)), \\
  (\text{Concert}, (S, B)), (\text{Concert}, (S, S)) \}.
  \]
- The player correspondence is \( P(\emptyset) = \{1\} \)
  and \( P(\text{Concert}) = \{1, 2\} \).
- The action correspondences are given by
  \( A_1(\emptyset) = \{\text{Book, Concert}\} \),
  \( A_1(\text{Concert}) = \{B, S\} \), and
  \( A_2(\text{Concert}) = \{B, S\} \).
- The preferences over terminal histories are
  given by the payoffs in the figure.
General Description

An extensive game with perfect information and simultaneous moves consists of:

- A set of players $I$.
- A set of terminal histories, where each terminal history is a sequence $(a^1, \ldots, a^k)$ in which each $a^\ell$ is a list of actions.
  - A proper subhistory is a proper initial segment of some terminal history.
- A player correspondence assigning a nonempty set $P(h) \subseteq I$ of players to each proper subhistory $h$.
- Action correspondences $A_i$ for each $i \in I$, where $A_i$ assigns a nonempty set of actions $A_i(h)$ to each proper subhistory $h$ with $i \in P(h)$.
- Preferences over terminal histories for each player.
Remarks:

- When players choose simultaneously, any list of actions must be possible.
  - If \( h = (a_1, \ldots, a^ \ell) \) is a proper partial history, and \( a \) is any list specifying an action \( a_i \in A_i(h) \) for each \( i \in P(h) \), then \( (a_1, \ldots, a^ \ell, a) \) is either a terminal history or a proper partial history.
  - *Osborne omits this condition from his description.*

- A normal form game is the special case of this structure in which the only proper subhistory is \( \emptyset \).

- An extensive game of perfect information is the special case of this structure in which each \( P(h) \) is a singleton (that is, a set with a single element).
Nash and Subgame Perfect Equilibrium

- There are extensive games with perfect information and simultaneous moves in which mixed strategies are involved in any sensible understanding of how the game should be played.

\[
\begin{bmatrix}
1 \\ 1
\end{bmatrix}
\]

- Even though this is the case, in Chapter 7, and these slides, we will focus attention on pure strategy equilibria.
• A strategy for a player $i$ is a function assigning an action in $A_i(h)$ to each proper partial history $h$ such that $i \in P(h)$.

• A Nash equilibrium is a profile of strategies $s$ such that no player can get a preferred outcome by switching to a different strategy.

• A subgame perfect equilibrium is a profile of strategies that restricts to a Nash equilibrium in every subgame (including the entire game).
  
  – The way to find the subgame perfect equilibria of an extensive game with perfect information and simultaneous moves is backwards induction: find the Nash equilibria of the games with only one (simultaneous) move, then work backwards through the tree.
  
  – The main difference from before is that now it will frequently be the case that there are multiple equilibria.
In the example we started with, the subgame after *Concert* has two pure Nash equilibria, namely \((B, B)\) and \((S, S)\).

Therefore the pure Nash equilibria of the entire game are \(((Book, S), S)\) and \(((Concert, B), B)\).
A slightly more complicated example illustrates how the number of subgame perfect equilibria tends to grow multiplicatively.

- The Nash equilibria of the subgame after $L$ are $(U, F)$ and $(D, B)$.
- The Nash equilibria of the subgame after $R$ are $(N, E)$ and $(S, W)$.
- For each pair of equilibria of the subgames there is a subgame perfect equilibrium, so the subgame perfect equilibria are:

$$(((L, U, N), (F, E)), ((L, U, S), (F, W))), \quad ((R, D, N), (B, E)), ((L, D, S), (B, W)).$$
**Example: Exit from a Declining Industry**

*Problem Description:*

- Initially there are two firms. In each period before it has exited the industry each firm must decide whether to exit.
  - If both firms are still in the industry, they make these decisions simultaneously.
- In each period it is in the industry firm $i$ produces $k_i$, where $k_1 > k_2 > 0$.
- The price in period $t$ is $p_t$ where $p_1 = P_t(k_1 + k_2)$ if both firms are in the industry, $p_t = P_t(k_1)$ if only firm 1 is in the industry, and $p_t = P_t(k_2)$ if only firm 2 is in the industry.
- Firm $i$’s profit in period $t$ is $(p_t - c)k_i$ if it is in the industry, and 0 otherwise.
- Each firm’s payoff for the game is the sum of it profits in all periods it is in the market.
Analysis:

- We assume that the numbers $P_t(k_1 + k_2)$, $P_t(k_1)$, and $P_t(k_2)$ are decreasing over time.

- Let $t_i$ be the last period that firm $i$ can make positive profits (i.e., the last period with $P_t(k_i) > c$) if it is the only firm in the industry.
  - In subgame perfect equilibrium firm $i$ will exit the industry after $t_i$ if not before.

- We assume that $t_2 > t_1$, and in fact that

  $$P_{t_1}(k_1 + k_2)(k_2 - c) + \sum_{t = t_1 + 1}^{t_2} P_t(k_2)(k_2 - c) > 0.$$  

  - Then Firm 2 will stay in the industry if both firms are still around in period $t_1$.
  - Knowing this, Firm 1 should exit in period $t_1$ if $P_{t_1}(k_1 + k_2) < c$. 
• Suppose that $P_{t_1-1}(k_1 + k_2) < c$.
  
  – Firm 2 will certainly stay, because the loss in this period, even if Firm 1 does not exit, is less than the loss if the two firms are still in next period, and Firm 2 was willing to endure that.
  
  – Since Firm 2 will stay, there is no point in Firm 1 not exiting.

• Continuing the backwards induction, we find that in subgame perfect equilibrium Firm 1 will exit in the first period $t$ such that $P_t(k_1 + k_2) < c$.

• This conclusion can be overturned if Firm 2 faces debt limits, since then Firm 1 may be able to force it into bankruptcy and then enjoy sole possession of the market until $t_1$. 
Voting by Binary Agenda

• A *binary tree* is a collection of terminal histories \((a_1, \ldots, a_k)\) where:
  - \(a_1, \ldots, a_k \in \{L, R\}\);
  - if \((a_1, \ldots, a_\ell)\) is a proper subhistory, then \((a_1, \ldots, a_\ell, L)\) and \((a_1, \ldots, a_\ell, R)\) are each either terminal histories or proper subhistories.

• A *binary agenda* consists of:
  - a finite binary tree;
  - a map that associates an outcome \(\alpha(h)\) with each terminal history of the tree.
• Given an odd number of players with strict preferences over outcomes, an extensive game with perfect information and simultaneous moves can be associated with a binary agenda:

  – For each nonterminal history \( h \), \( P(h) \) is the set of all players, and for each player \( i \), \( A_i(h) = \{L, R\} \).
    
  • A history is then a sequence \( h = (v^1, \ldots, v^k) \) in which each \( v^j \) is a list of all players’ votes in \( \{L, R\} \).

  • For each such history \( h \) let \( a(h) = (a_1, \ldots, a_k) \) be the list in which each \( a_j \in \{L, R\} \) is the recipient of the most votes in \( v^j \).

  – The terminal histories are the sequences \( (v^1, \ldots, v^k) \) such that \( a(h) \) is a terminal history of the binary tree.

  – Player \( i \) prefers terminal history \( h \) to terminal history \( h' \) if she prefers the resulting outcome \( \alpha(a(h)) \) to \( \alpha(a(h')) \).
Sophisticated Voting

Analysis:

• There are *lots* of subgame perfect equilibria.
  – For example, any profile of strategies such that no single player can ever affect the outcome if the strategies are followed.

• We will assume *sophisticated voting*: each player votes for the branch of the tree whose outcome (under sophisticated voting) they prefer.
  – Backwards induction can be used to prove that sophisticated voting is a well defined concept: for an odd number of voters with given preferences, each binary agenda has a uniquely defined outcome.
Condorcet Winners

- An outcome $x$ is a *Condorcet winner* if, for each other outcome $y$, a majority prefer $x$ to $y$.

- If there is a Condorcet winner, it will be the outcome of sophisticated voting for any binary agenda.
  - The proof is by backwards induction: at any point where the choice is between two branches, one of which leads to the Condorcet winner under sophisticated voting, sophisticated voting will choose that branch.

- It can easily happen that there is no Condorcet winner.
The Top Cycle Set

An outcome $x$ is in the top cycle set if, for any other outcome $y$, there is a sequence $u_1, \ldots, u_k$ such that a majority prefer $x$ to $u_1$, a majority prefer $u_j$ to $u_{j+1}$ for $j = 1, \ldots, k + 1$, and a majority prefer $u_k$ to $y$.

- The sophisticated voting outcome, say $x$, is always in the top cycle set.
  - For any other outcome $y$ the tree contains a point where sophisticated voters choose $x$ over the outcome $u_1$ that would result if they went down a branch leading toward $y$, another point where they choose $u_1$ to the outcome $u_2$ that would result if they continued down the branch leading toward $y$, and so forth.
  - This proves that the top cycle set is nonempty!
For any element $x$ of the top cycle set there is a binary agenda that gives $x$.

- The figure illustrates the construction, showing how to craft an agenda when majorities prefer $x$ to $u_1$, $u_1$ to $u_2$, $u_2$ to $y$, $x$ to $v_1$, $v_1$ to $v_2$, and $v_2$ to $z$. 

\begin{center}
\begin{tikzpicture}
  \node (y) at (0,0) {$y$};
  \node (u1) at (1,-1) {$u_1$};
  \node (u2) at (-1,-1) {$u_2$};
  \node (z) at (0,-2) {$z$};
  \node (x) at (2,-2) {$x$};
  \node (v1) at (1,-3) {$v_1$};
  \node (v2) at (-1,-3) {$v_2$};

  \draw[-latex] (y) -- (u1);
  \draw[-latex] (u2) -- (u1);
  \draw[-latex] (u1) -- (x);
  \draw[-latex] (y) -- (z);
  \draw[-latex] (z) -- (v1);
  \draw[-latex] (u2) -- (v1);
  \draw[-latex] (v1) -- (v2);
  \draw[-latex] (v2) -- (x);
\end{tikzpicture}
\end{center}
**Chance Moves**

The model can be extended to incorporate random events, such as rolls of dice.

- One introduces a new player, sometimes called *Chance* or *Nature* that moves randomly according to specified probabilities.

- The preferences of the players (other than Chance, which has no preferences) have to be over lotteries over terminal history.

- The subgame perfect equilibria are still computed using backwards induction, but it takes more numerical computation.

```
A  1

[1]   B
[1]  1/2

2

C  [0]  D  [1]
[1]   1/2

3

[3]
[0]
```
Example: Sequential Duel

Problem Description:

- In odd numbered periods player 1 may choose to shoot at player 2, and in even numbered periods player 2 may choose to shoot at player 1.
  - The probability that a shot at player j kills her is $p_i$, where $0 < p_i < 1$.
- If a player is killed the game ends.
- There are two variants:
  - Finite horizon: the game ends after period $T$, even if both players are alive.
  - Infinite horizon: the game continues until one player is killed, or forever.
- Each player is primarily concerned with her own survival.
- Conditional on surviving herself, each player may prefer that the opponent survive, or each may prefer that the opponent die.
Analysis:

• If the finite horizon case backwards induction gives a unique solution:
  – If each player would like the other to survive, there will never be any shooting.
  – If each player would like the other to die, each player will shoot at every opportunity.

• In the infinite horizon case backwards induction does not work.

• Always shooting is a subgame perfect equilibrium even when each player would prefer that both of them survive.

• There is always a subgame perfect Nash equilibrium in which both play the grim trigger strategy:
  – shoot if any shots were fired in the past;
  – don’t shoot first.

• There are many other equilibria.