Economics 3012
Strategic Behavior
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September 8, 2006

Lecture 7

Topics

• Problem Set 6

• Two Problems on Mixed Equilibrium

• Applications of Games of Perfect Information
  – The Ultimatum Game.
  – The Holdup Game.
  – Stackleberg Duopoly.
  – Buying Votes.
  – A Race.
**Problem 1:** Do Exercise 163.2 of Osborne.

*Problem Statement:* There are three possible policies $X$, $Y$, and $Z$. Player 1 vetoes one of them, then Player 2 vetoes one of the two remaining policies. The policy that is left at the end is implemented. The preferences are

$$X \succ_1 Y \succ_1 Z \quad \text{and} \quad Z \succ_2 Y \succ_2 X.$$ 

Model this as an extensive game and find its Nash equilibria.

*Analysis:* The extensive form of this game (with outcomes rather than payoffs at the terminal nodes) is:

![Tree Diagram](image.png)
The mechanical method of computing the set of Nash equilibria is to pass to the normal form:

\[
\begin{pmatrix}
 v_X & v_Y & v_Z \\
 v_Y v_X v_X & Z & Z & Y \\
 v_Y v_X v_Y & Z & Z & X \\
 v_Y v_Z v_X & Z & X & Y \\
 v_Y v_Z v_Y & Z & X & X \\
 v_Z v_X v_X & Y & Z & Y \\
 v_Z v_X v_Y & Y & Z & X \\
 v_Z v_Z v_X & Y & X & Y \\
 v_Z v_Z v_Y & Y & X & X \\
\end{pmatrix}
\]

(To display it on the page we have made player 2 the row player.)

- There is no Nash equilibrium in which player 1 plays \( v_X \).
  - Player 2 would have to be playing one of the pure strategies that brings about \( Z \).
  - This would imply that player 1’s action was not a best response to player 2’s action.
• The same reasoning shows that there cannot be a Nash equilibrium in which player 1 plays \( v_Y \).

• If player 1 plays \( v_Z \), player 2’s best responses are the four pure strategies that bring about \( Y \).

• In order for this to be a Nash equilibrium, it must also be the case that player 1 cannot switch to bring about \( X \).

• Thus there are two possibilities for Nash equilibria, namely

\[
(v_Z, v_Y v_X v_X) \text{ and } (v_Z, v_Z v_X v_X).
\]

– It is easily verified that, in fact, these are equilibria.
A more sophisticated approach is to recognize that in an extensive form game of perfect information in which, along each path through the tree, no player moves more than once, a Nash equilibrium is a pair of pure strategies with the property that behavior at every point on the “path” (that is, the part of the tree that occurs when the equilibrium is played) is rational, given what is happening both on the path and off it.

- Since player 1 can prevent $Z$, and any strategy other than $v_Z$ allows player 2 to bring about $Z$, in any Nash equilibrium player 1 must play $v_Z$ after which player 2 brings about $Y$.

- The remaining condition that must be satisfied is that player 2’s behavior off the path must be such that player 1 cannot bring about $X$ by deviating.
  - Therefore player 2 must be playing $v_X$ in response to $v_Y$ by player 1.
Problem 2: Do Exercise 173.3 of Osborne.

Problem Statement: Find the subgame perfect equilibrium of the game in Exercise 163.2. Are there Nash equilibria that are not subgame perfect, and do any such Nash equilibrium generate a different outcome? If we vary the preferences of the agents, can it happen that the subgame perfect equilibrium outcome depends on which agent has the first veto?

Analysis: The subgame perfect Nash equilibrium is shown below.
• There is another Nash equilibrium, as we saw in Exercise 163.2.
  – It also gives the outcome $Y$.

• To get a variant in which the outcome depends on which player vetoes first we assume that the preferences are

$$X \succ_1 Y \succ_1 Z \quad \text{and} \quad Y \succ_2 X \succ_2 Z.$$  

• The subgame perfect equilibria are shown in the two diagrams below.

• Whichever player goes first gets her favorite alternative by vetoing the other player’s favorite alternative, knowing that the other player will then veto $Z$. 
Problem 3: Do Exercise 173.4 of Osborne.

Problem Statement: Army 1 must decide whether to attack an island held by Army 2. If Army 1 decides to attack, Army 2 must decide whether to fight or retreat across a bridge to the mainland. Army 1 prefers attacking followed by a retreat of Army 2 to not attacking, but not attacking is better than a fight. Army 2 prefers not being attacked to retreating and retreating to fighting. Show that Army 2 would be better off, in subgame perfect equilibrium, if they burnt the bridge to the mainland.

Analysis: The figures below show the subgame perfect equilibria of the game with the bridge and without.
We can also combine these into a single game with the decision about whether to burn at the beginning:
Exercise 130.3 of Osborne

Problem Statement: There are two players. Each player chooses a demand $d_i$, which is an even number between 0 and 10. If $d_1 + d_2 \leq 10$, then player $i$ receives

$$d_i + \frac{1}{2}(10 - (d_1 + d_2)) = 5 + \frac{1}{2}(d_i - d_j).$$

If $d_1 + d_2 > 10$, then both players receive 0.

Find all the symmetric equilibria in which each player mixes over two demands.

Analysis: To clarify the problem we can display it in normal form:

$$
\begin{array}{ccccccc}
0 & 2 & 4 & 6 & 8 & 10 \\
0 & (5,5) & (4,6) & (3,7) & (2,8) & (1,9) & (10,0) \\
2 & (6,4) & (5,5) & (4,6) & (3,7) & (2,8) & (0,0) \\
4 & (7,3) & (6,4) & (5,5) & (4,6) & (0,0) & (0,0) \\
6 & (8,2) & (7,3) & (6,4) & (0,0) & (0,0) & (0,0) \\
8 & (9,1) & (8,2) & (0,0) & (0,0) & (0,0) & (0,0) \\
10 & (10,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0)
\end{array}
$$
• If player 1 is assigning positive probability to \(d_1\) and \(e_1\), where \(d_1 < e_1\), then the two demands that make sense for player 2 are

\[d_2 = 10 - e_1 \text{ and } e_2 = 10 - d_1.\]

– A demand below \(d_2\) always does worse than \(d_2\).
– A demand strictly between \(d_2\) and \(e_2\) succeeds only when \(e_2\) would succeed, and gives less when it does.
– A demand above \(e_2\) always gives 0.

• In the following calculations it is useful to note that, when \(d_2 = 10 - e_1\), we have

\[d_1 + \frac{1}{2}(10 - (d_1 + d_2)) = \frac{1}{2}(d_1 + e_1).\]
• Let \( q \) be the probability that player 2 chooses \( e_2 \).

• Since player 1 is indifferent between \( d_1 \) and \( e_1 \), it must be the case that

\[
(1 - q) \frac{1}{2}(d_1 + e_1) + qd_1 = (1 - q)e_1.
\]

• Solving for \( q \), we find that \( q = \frac{e_1 - d_1}{e_1 + d_1} \).

• Summarizing, if \( d_2 = 10 - e_1 \) and \( e_2 = 10 - d_1 \), then there is a Nash equilibrium

\[
((1 - p)d_1 + pe_1, (1 - q)d_2 + qe_2)
\]

where \( p = \frac{e_2 - d_2}{e_2 + d_2} \) and \( q = \frac{e_1 - d_1}{e_1 + d_1} \).

• This is more than the problem asks us to do. To get the symmetric equilibria of this sort we specialize to the case \( d_1 = d_2 \) and \( e_1 = e_2 \).
Exercise 141.1 of Osborne

Problem Statement: Find all the mixed strategy Nash equilibria of the following game:

\[
\begin{array}{ccc}
L & M & R \\
T & (2,2) & (0,3) & (1,3) \\
B & (3,2) & (1,1) & (0,2)
\end{array}
\]

Analysis:

- Both \(L\) and \(M\) are weakly dominated by \(R\).
- If player 1 mixes, then:
  - \(R\) is player 2’s unique best response;
  - \(T\) is player 1’s only best response to \(R\).
    * That is, player 1 is not mixing.
  - Therefore all equilibria have player 1 playing a pure strategy.
• If player 1 plays $T$, player 2 is indifferent between $M$ and $R$, and in equilibrium can play any mixture that makes $T$ a best response for player 1.
  
  – This means that the probability of $R$ must be at least $1/2$.

• If player 1 plays $B$, then player 2 is indifferent between $L$ and $R$, and in equilibrium can play any mixture that makes $B$ a best response for player 1.
  
  – This means that the probability of $L$ must be at least $1/2$.

Thus the set of equilibria is

$$\{ (T,(1 - \alpha)M + \alpha R) : \frac{1}{2} \leq \alpha \leq 1 \} $$

$$\cup \{ (B,(1 - \beta)L + \beta R) : 0 \leq \beta \leq \frac{1}{2} \}.$$


The Ultimatum Game

The rules are as follows:

• Player 1 chooses a number $x$ between 0 and $c$.

• After seeing $x$ player 2 chooses either to “accept” the offer, in which case the payoffs are $(c - x, x)$, or “reject” the offer, in which case the payoffs are $(0, 0)$.

The unique subgame perfect equilibrium of this game has:

• Player 1 choosing $x = 0$.

• Player 2 accepting.

• We think of this as a “limiting simplification” of player 1 offering some small $\varepsilon > 0$ and player 2 accepting because something is better than nothing.

  – *This style of subgame perfect equilibrium occurs frequently.*

There are *lots* of Nash equilibria that are not subgame perfect.
• There has been a great deal of experimental work on the Ultimatum Game.
  – It is important to distinguish between the “monetary” and “utility” versions of the game.
  * Experimental work is necessarily restricted to the monetary version.

• In general, the findings are that:
  – “Unfair” offers are frequently rejected, even though this is (from a purely monetary point of view) irrational.
  – Either anticipating this, or because they prefer behaving fairly, offers significantly greater than 0 are common.
  – These effects are observed even when the amounts of money are on the order of a month’s income.
The Holdup Game

• In the Holdup Game the Ultimatum Game is preceeded by a stage in which the second player decides how much to invest:
  – Player 2 chooses an investment level \( I \in \{L, H\} \), where \( L < H \).
    * The investment determines \( c = c_I \), where \( c_L < c_H \).
  – After seeing \( I \), Player 1 chooses a number \( x \) between 0 and \( c \).
  – After seeing \( x \), Player 2 chooses either to “accept,” giving payoffs \((c - x, x - I)\), or “reject,” giving payoffs \((0, -I)\).

• Subgame perfect equilibrium:
  – In both subgames after \( I \) is chosen player 1 offers 0, which is accepted.
  – Anticipating this, player 2 chooses \( L \).

“Holdup” is important when one firm supplies a specialized input to another firm.
Stackleberg Duopoly

Stackleberg duopoly is like Cournot duopoly, insofar as the firms are choosing quantities, but now one firm (the “leader”) chooses first and the second firm (the “follower”) responds.

- Firm 1 chooses $q_1$.
- After seeing $q_1$, Firm 2 chooses $q_2$.
- The payoff to firm $i$ is

\[ q_i P(q_1 + q_2) - C_i(q_i). \]

Subgame perfect equilibrium:

- Suppose that, for each $q_1$, Firm 2 has a unique best response $b_2(q_1)$.
- Firm 1 will then choose $q_1$ to maximize

\[ q_1 P(q_1 + B_2(q_1)) - C_1(q_1). \]
Examples:

- Suppose that
  \[ P(Q) = \begin{cases} 
  \alpha - Q, & Q \leq \alpha, \\
  0, & Q > \alpha 
  \end{cases} \]
  and, for some \( c > 0 \),
  \[ C_i(q_i) = cq_i. \]

- Setting
  \[ 0 = \frac{d[q_2(\alpha-q_1-q_2)-cq_2]}{dq_2} = (\alpha - c - q_1) - 2q_2 \]
  shows that
  \[ b_2(q_1) = \begin{cases} 
  \frac{\alpha-c-q_1}{2}, & q_1 < \alpha - c, \\
  0, & \text{otherwise}. 
  \end{cases} \]

- Firm 1 maximizes
  \[ q_1(\alpha - q_1 - \frac{\alpha-c-q_1}{2}) - cq_1 = \frac{1}{2} q_1 (\alpha - c - q_1) \]
  by setting \( q_1 = \frac{\alpha-c}{2} \), with the result that
  \[ q_2 = \frac{\alpha-c}{4}. \]
Remarks about Stackleberg Duopoly:

- Firm 1 is better off than Firm 2.
  - In general, Firm 1 has the option of choosing the Cournot output, after which Firm 2’s best response would be the Cournot output, and the two firms’ profits would (assuming symmetric costs) be equal. But Firm 1 may be able to do better.
  - More directly, when Firm 1 increases output there is the “added bonus” that Firm 2 produces less.

- Total output is greater than in the Cournot case, but less than in the Bertrand/competitive case.
Buying Votes

- There are two possible policies X and Y.
- Two interests groups compete to implement their favorite of the two.
- There are an odd number $k$ of legislators.
  - First the supporters of X choose $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$.
  - Then the supporters of Y choose $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$.
  - Each legislator $j$ supports X if $x_j > y_j$, and supports Y otherwise.
  - The proposal with majority support is implemented.
- The payoff for the supporters of X is
  \[
  \begin{cases} 
  V_X - (x_1 + \cdots + x_k) & \text{if } X \text{ passes}, \\ 
  -(x_1 + \cdots + x_k) & \text{if } Y \text{ passes}, 
  \end{cases}
  \]
  and similarly for the supporters of Y.
Analysis

In the case \( k = 3 \) and \( V_X = V_Y = 1 \) this is the Chopstick Auction, except that now the players bid in turn.

- Our analysis showing that the Chopstick Auction has no pure strategy equilibria works here to show that the supporters of \( X \) cannot win the issue without paying more than it is worth.

In general:

- The supporters of \( X \) can win, profitably, if they can choose \( x \) such that it is not worthwhile for the supporters of \( Y \) to outbid them for \( \mu = \frac{1}{2}(k + 1) \) legislators.
  - Therefore the supporters of \( X \) will offer all legislators the same amount.

- Making \( X \) win takes \( kV_Y/\mu \).

- Thus the supporters of \( X \) will win if \( \mu V_X > kV_Y \), and they will not contest the issue if \( \mu V_X < kV_Y \).
A Race

Two people are racing to be the first to reach some goal.

- For player $i = 1, 2$, being the first to reach the goal is worth $v_i$.

- At the beginning of the game player $i$ is $k_i$ steps from the goal.

- The players take turns. At a player’s turn she chooses to take zero, one, or two steps toward the goal.
  - If the two players choose zero steps in succession the game ends with neither player reaching the goal.
  - Let $0 = c(0) < c(1) < c(2)$ be the costs of moving zero, one, or two steps in a period.
• Let \((x_1, \ldots, x_{T_1}, y_1, \ldots, y_{T_2})\) be the sequences of moves of the two players in a complete play.
  
  – Depending on who goes first and how the game ends, \(T_2\) may be \(T_1 - 1\), \(T_1\), or \(T_1 + 1\).

• If player 1 reaches the goal first, then the payoffs are
  
  \[
  (v_1 - \sum_{t=1}^{T_1} c(x_t), -\sum_{t=1}^{T_2} c(y_t)).
  \]

• If player 1 reaches the goal first, then the payoffs are
  
  \[
  (\sum_{t=1}^{T_1} c(x_t), v_2 - \sum_{t=1}^{T_2} c(y_t)).
  \]

• If neither player reaches the goal, then the payoffs are
  
  \[
  (\sum_{t=1}^{T_1} c(x_t), -\sum_{t=1}^{T_2} c(y_t)).
  \]
Analysis: We study the subgame perfect equilibrium of this game, which we will assume is unique.

- If a player has no way to reach the goal profitably, she will simply choose zero steps in each period.
- If one player has a way to reach the goal profitably, and the other does not, then the first player will follow a least-cost path to the goal.
- In a subgame perfect equilibrium, only one player will actually incur costs.
- We say that the game is lost if, in the subgame perfect equilibrium, the first move is zero steps.
  - Either:
    * the game is lost, or
    * there is a way to move that creates a lost subgame for the opponent.
  - It can happen that both are the case, if winning isn’t worth it.
The Case Studied by Osborne

• We consider the parameters studied by Osborne: $6 < v_1, v_2 < 7$, $c(1) = 1$, and $c(2) = 4$.

• We will fill in a $6 \times 6$ matrix of pairs, where:
  – The row number is $k_1$, and the column number is $k_2$.
  – The first entry in the $(k_1, k_2)$ pair is the status of the game when player 1 has the first move, and the second entry is the status when player 2 has the first move.
  – The possible statuses are:
    * $l$: the player to move loses;
    * $w$: the player to move wins by moving one step;
    * $j$: the player to move wins by “jumping”, i.e., moving two steps.
We begin with a blank matrix:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
2 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
3 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
4 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
5 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
6 & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
\end{pmatrix}
\]

A player one step from the finish line will take that step:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (w,w) & (w,*) & (w,*) & (w,*) & (w,*) \\
2 & (*,w) & (*,*) & (*,*) & (*,*) & (*,*) \\
3 & (*,w) & (*,*) & (*,*) & (*,*) & (*,*) \\
4 & (*,w) & (*,*) & (*,*) & (*,*) & (*,*) \\
5 & (*,w) & (*,*) & (*,*) & (*,*) & (*,*) \\
6 & (*,w) & (*,*) & (*,*) & (*,*) & (*,*) \\
\end{pmatrix}
\]
If the other player is one step away, you are lost unless you win right away:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (w,w) & (w,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
2 & (j,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
3 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
4 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
5 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
6 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
\end{array}
\]

When \((k_1, k_2) = (2, 2)\), jumping is better than letting the opponent win:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (w,w) & (w,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
2 & (j,w) & (j,j) & (*,*) & (*,*) & (*,*) & (*,*) \\
3 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
4 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
5 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
6 & (l,w) & (*,*) & (*,*) & (*,*) & (*,*) & (*,*) \\
\end{array}
\]
If you are two steps away, and you are ahead, just take one step:

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If the other player is two steps away, and you are behind, you are lost:

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If you are three steps away, and not behind, take one step:

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If the opponent is three steps away, and you are behind, jump or lose:

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<tr>
<td>1</td>
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</table>
When \((k_1, k_2) = (4, 4)\), jumping stops the opponent from jumping:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (w,w) & (w,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
2 & (j,w) & (j,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
3 & (l,w) & (l,w) & (w,w) & (w,j) & (w,l) & (w,l) \\
4 & (l,w) & (l,w) & (j,w) & (j,j) & (\ast,\ast) & (\ast,\ast) \\
5 & (l,w) & (l,w) & (l,w) & (\ast,\ast) & (\ast,\ast) & (\ast,\ast) \\
6 & (l,w) & (l,w) & (l,w) & (\ast,\ast) & (\ast,\ast) & (\ast,\ast) \\
\end{array}
\]

If you are four steps away, and ahead, take one step:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & (w,w) & (w,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
2 & (j,w) & (j,j) & (w,l) & (w,l) & (w,l) & (w,l) \\
3 & (l,w) & (l,w) & (w,w) & (w,j) & (w,l) & (w,l) \\
4 & (l,w) & (l,w) & (j,w) & (j,j) & (w,\ast) & (w,\ast) \\
5 & (l,w) & (l,w) & (l,w) & (\ast,w) & (\ast,\ast) & (\ast,\ast) \\
6 & (l,w) & (l,w) & (l,w) & (\ast,w) & (\ast,\ast) & (\ast,\ast) \\
\end{array}
\]
If the opponent is four steps away, and you are behind, jumping doesn’t work:

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For the rest, the player who is ahead, or the first mover, wins (jumping isn’t worth it):

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