Economics 3012
Strategic Behavior
Andy McLennan
August 25, 2006

Lecture 5

Topics

• Problem Set 4

• Examples of Mixed Nash Equilibrium

• Finding All Equilibria
Problem Set 4

Exercise 118.2

*Problem Description:*

- A voter receives:
  - 2 utils if her favorite candidate wins,
  - 1 util in the event of a tie,
  - 0 utils if her favorite candidate loses.
- The cost of voting is $c$ where $0 < c < 1$.
- Candidate $A$ has $k$ supporters.
- Candidate $B$ has $m \geq k$ supporters.

We are looking for a mixed strategy Nash equilibrium in which:

- every supporter of Candidate $A$ votes with probability $p$;
- $k$ supporters of Candidate $B$ vote with certainty;
- the remaining $m - k$ supporters of Candidate $B$ abstain.
**Analysis:**

- In general a voter is *pivotal* if her vote affects the outcome, either:
  - between a loss and and a tie or
  - between a tie and a win.

- The two different ways of being pivotal each have value one.

- Let $p$ be the probability that a supporter of Candidate $A$ votes.

- The probability that a supporter of Candidate $A$ is pivotal is just the probability $p^{k-1}$ that the other $k-1$ all are voting.

- A supporter of candidate $A$ is indifferent between voting and abstaining if the probability $p^{k-1}$ of being pivotal is equal to the cost $c$ of voting, so
  \[ p = c^{\frac{1}{k-1}}. \]
• For a supporter of Candidate B who is expected to vote the probability of being pivotal is the sum of:
  – the probability \( p^k = c^{\frac{k}{k-1}} \) that all \( k \) supporters of Candidate A vote;
  – the probability \( kp^{k-1}(1 - p) \) that exactly \( k - 1 \) of them vote.

• This is greater than the probability \( p^{k-1} = c \) that a particular group of \( k - 1 \) of supporters of A.
  – Thus all the supporters of Candidate B who are expected to vote strictly prefer to do so.

• A supporter of Candidate B who is not expected to vote is pivotal with probability
  \[
p^k = c^{\frac{k}{k-1}} = c \cdot c^{\frac{1}{k-1}} < c.
  \]
  – Such a person prefers not to vote.

• The expected number of voters \( pk + k = (c^{\frac{1}{k-1}} + 1)k \) increases with \( c \).
Exercise 118.3

Problem Statement:

• General A, with three divisions, and General B, with two divisions, each have to allocate their forces to two passes.

• For General A a pure strategy is given by the number $d_1 \in \{0, 1, 2, 3\}$ of divisions allocated to the first pass.

• A pure strategy for General B is given by the number $d_2 \in \{0, 1, 2\}$ of divisions allocated to the first pass.

• General A wins if and only if $d_1 \geq d_2$ and $3 - d_1 \geq 2 - d_2$. 
Analysis:

- For General A:
  - $d_1 = 0$ is weakly dominated by $d_1 = 1$.
  - $d_1 = 3$ is weakly dominated by $d_1 = 2$.

- There cannot be a Nash equilibrium in which General B assigns positive probability to $d_2 = 1$:
  - General A would never choose $d_1 = 0, 3$, so $d_2 = 1$ would lose for sure.
  - But General B can always get a positive expected payoff.

- In any equilibrium General B assigns probability 1/2 to each of $d_2 = 0$ and $d_2 = 2$, because if General B assigned unequal probabilities:
  - General A would assign two divisions to whichever pass General B most frequently assigned two divisions to.
  - The best response to this would be for General B to assign both divisions to the other pass.
• In order for General $B$ to be indifferent between $d_2 = 0$ and $d_2 = 2$, the sum of probabilities assigned to $d_1 = 0$ and $d_1 = 1$ must be equal to the sum of probabilities assigned to $d_1 = 2$ and $d_1 = 3$.

• In order for General $B$ to not prefer $d_2 = 1$ is must be the case that the sum of the probabilities assigned to $d_1 = 0$ and $d_1 = 3$ must not be greater than $1/2$. 
Exercise 128.1

Problem Statement:

- Two consumers simultaneously choose which store to go to.
  - Each trades with probability 1/2 if they both go to the same store.
  - Trading at price $p$ results in a utility of $1 - p$.
  - Not trading gives utility 0.

- By symmetry it suffices to consider prices $p_1, p_2$ with $p_1 \leq p_2$, so we can call the first seller the Discount House and the second seller the Boutique.

- We have the following payoff matrix:

$$
D_1 \begin{pmatrix} \frac{1-p_1}{2}, \frac{1-p_1}{2} \\ (1 - p_2, 1 - p_1) \end{pmatrix} \quad B_1 \begin{pmatrix} (1 - p_1, 1 - p_2) \\ \frac{1-p_2}{2}, \frac{1-p_2}{2} \end{pmatrix}.
$$
Analysis:

• It is certainly better to go to the Discount House if the other buyer goes to the Boutique.
  – Going to the Discount House will be a dominant strategy, and \((D_1, D_2)\) will be the unique Nash equilibrium, if 
    \[1 - \frac{p_1}{2} \geq 1 - p_2,\]  
    i.e.,  
    \[p_1 \leq 2p_2 - 1.\]
• If $\frac{1-p_1}{2} < 1 - p_2$, then the game is a Battle of the Sexes with:
  - two pure equilibria $(D_1, B_2)$ and $(B_1, D_2)$
  - a mixed equilibrium.

• Let $q$ be the probability that Buyer 2 goes to the Discount House in the mixed equilibrium. Indifference for Buyer 1 implies that

$$q \frac{1-p_1}{2} + (1-q)(1-p_1) = q(1-p_2) + (1-q)\frac{1-p_2}{2}.$$  

• Solving this gives

$$q = \frac{1 - 2p_1 + p_2}{2 - p_1 - p_2}.$$  

• Since the game is symmetric with respect to interchange of the two buyers, this is also the probability that Buyer 1 goes to the Discount House.
Reporting a Crime

Problem Description:

- Each of \( n \) people witness a crime. For each of them reporting the crime has a cost of \( c \), and each receives a benefit of \( v \) if someone reports the crime:
  - \( S_i = \{R_i, D_i\}, i = 1, \ldots, n; \)
  - for each \( i \), \( u_i(s) \) is:
    * 0 if \( s_j = D_j \) for all \( j \);
    * \( v \) if \( s_j = R_j \) for some \( j \neq i \) and \( s_i = D_i \);
    * \( v - c \) if \( s_i = R_i \).
Analysis:

- For each player, there is a pure Nash equilibrium in which that person reports the crime and no one else does.

- There is an equilibrium in which each reports with probability

  \[ p_n = 1 - (c/v)^{n-1} \]

  because then, for each the expected utilities of the two actions are the same:

  \[ v - c = (1 - (1 - p_n)^{n-1})v. \]

- Since \((1 - p_n)^n = (c/v)^{n-1}\) is an increasing fraction of \(n\), the probability that no one calls in this particular equilibrium increases with \(n\).
• If one player, say $i$, reports more frequently than another, say $j$, then the probability that someone reports if $i$ does not is lower than the probability that someone reports if $j$ does not.
  – In a mixed Nash equilibrium in which two players $i$ and $j$ are mixing, they must mix with the same probability.
  – For any $k = 1, \ldots, n$ and any group of $k$ agents, there is a mixed Nash equilibrium in which each member of the group reports with probability $p_k$ and those outside the group never report.

• Remarks on the Kitty Genovese case:
  – $(c/v)^{n-1} < c/v$ for all $n$.
  – What determines the size $k$ of the equilibrium group of mixers?
The Condorcet Jury Theorem

Problem Description:

- The Accused is either guilty \((G)\) or innocent \((I)\).

- Each juror \(i = 1, \ldots, n\) observes a signal \(\sigma_i \in \{i, g\}\).

  \[
  \text{Prob}(\sigma_i = g|G) = \text{Prob}(\sigma_i = i|I) = p > \frac{1}{2}.
  \]

  - Conditional on the actual state \((G\) or \(I)\) the signals are statistically independent.

- Each juror chooses a vote \(s_i \in \{G_i, I_i\}\).

- The Accused is convicted if \(s_i = G_i\) for all \(i\) and acquitted otherwise.

- For each juror the payoff is:
  
  - 0 if the correct verdict is reached;
  
  - \(-1\) if a criminal is acquitted;
  
  - \(-L\) if an innocent is convicted.
Analysis:

• There are many equilibria in which no one is pivotal. Our analysis is not exhaustive.

• *Sincere voting* is the strategy of choosing $G_i$ when $\sigma_i = g$ and $I_i$ when $\sigma_i = i$.

• When $n$ is large, *sincere voting is not a Nash equilibrium*.
  
  – Juror $i$’s vote is *pivotal* only when all other jurors are voting for guilt.
  
  – If everyone else is voting sincerely the net gain for $i$ resulting from switching to always voting for guilt is:
    \[
    \frac{1}{2}p^{n-1}(1-p) \times 1 + \frac{1}{2}(1-p)^n \times (-L).
    \]
    * $p^{n-1} > (1-p)^{n-1}L$ when $n$ is large.

• It turns out that the probability of convicting an innocent in the symmetric Nash equilibrium in which each juror $i$ votes for guilt when $\sigma_i = g$ and mixes when $\sigma_i = i$ is bounded away from 0 as $n \to \infty$. 
A Characterization of Equilibrium

- The support of $\alpha_i \in \Delta(S_i)$ is the set of $s_i$ such that $\alpha_i(s_i) > 0$.

- The support of a mixed strategy profile $\alpha$ is the $n$-tuple $(T_1, \ldots, T_n)$ where, for each $i$, $T_i$ is the support of $\alpha_i$.

- Application of the distributive law to the formula defining expected payoffs gives the formula

$$u_i(\alpha) = \sum_{s_i \in S_i} \alpha_i(s_i)u_i(s_1, \alpha_{-i}).$$

Proposition: A mixed strategy profile $\alpha^*$ is a Nash equilibrium if and only if it is the case for each $i$ that:

- $u_i(s_i, \alpha^*_{-i}) = u_i(t_i, \alpha^*_{-i})$ for all $s_i$ and $t_i$ in the support of $\alpha_i^*$, and

- $u_i(s_i, \alpha^*_{-i}) \geq u_i(t_i, \alpha^*_{-i})$ for all $s_i$ in the support of $\alpha_i^*$ and all $t_i$ outside the support of $\alpha_i^*$. 
• A strict Nash equilibrium is a Nash equilibrium $\alpha^*$ such that $u_i(s_i, \alpha^*_{-i}) > u_i(t_i, \alpha^*_{-i})$ for all $s_i$ in the support of $\alpha^*_i$ and all $t_i$ outside the support of $\alpha^*_i$.

• A totally mixed Nash equilibrium is a Nash equilibrium $\alpha^*$ such that for each $i$ the support of $\alpha^*_i$ is all of $S_i$.

• A totally mixed Nash equilibrium is strict, obviously.

• Any Nash equilibrium $\alpha^*$ is (in the obvious sense) a totally mixed equilibrium of the game obtained by removing all the pure strategies in $S_i$ that are not in the support of $\alpha^*_i$ for each $i$. 
A Recipe for Finding All Nash Equilibria

The following will always work ...

- List all the $n$-tuples $(T_1, \ldots, T_n)$ where $\emptyset \neq T_i \subset S_i$.

- For each $(T_1, \ldots, T_n)$ find all the totally mixed Nash equilibria of the truncated game obtained by eliminating all elements of each $S_i \setminus T_i$.
  - The totally mixed Nash equilibria are the solutions of a system of equations that are nonlinear when $n \geq 3$.

- For each totally mixed equilibrium of the truncated game, ask if there is an agent $i$ who can get a higher payoff by switching to some pure strategy in $S_i \setminus T_i$.
  - If the answer is “no” you’ve found an equilibrium.
... but it’s *extremely* tedious.

- For each $i$ there are $2^{|S_i|} - 1$ nonempty subsets of $S_i$. For example
  \[
  \prod_{i=1}^{n} (2^{|S_i|} - 1) = 49
  \]
  when $n = 2$ and $|S_1| = |S_2| = 3$, and this number quickly gets *much* bigger as the game expands.

- When $n > 2$, a single $(T_1, \ldots, T_n)$ can have *lots* of totally mixed Nash equilibria.
  - When 8 players each have 2 pure strategies, there can be up to 14,833 totally mixed equilibria.
  - When 6 players each have 6 pure strategies, there can be up to $4.1 \times 10^{17}$ totally mixed equilibria.
What to do about this?

- It is *not* unusual for games to have many equilibria.
  - When $n = 2$, each player has $k$ pure strategies, and the payoffs $u_i(s)$ are independent identically distributed normal random variables, the mean number of Nash equilibria is about $1.32^k$.
  - When $n = 6$, each player has 6 pure strategies, and the payoffs $u_i(s)$ are independent identically distributed normal random variables, the mean number of Nash equilibria is around $5.6 \times 10^6$.

- Inevitably, one must, to some extent, restrict attention to:
  - small games;
  - games with special structures.

- For these there are some speed-ups...
Symmetric Games

• Sometimes a game is “unchanged” if one interchanges some strategies.
  - In “Penalty Kick” there is a symmetry $L_1 \rightarrow R_1, R_1 \rightarrow L_1, L_2 \rightarrow R_2, R_2 \rightarrow L_2$.

• A more comprehensive notion of symmetry involves interchanging the players as well as the pure strategies.
  - For “Which Side to Drive On” there is a symmetry that interchanges the two agents, sending $L_1 \rightarrow R_2, R_1 \rightarrow L_1, L_2 \rightarrow R_1, \text{ and } R_2 \rightarrow L_1$.

\[
\begin{pmatrix}
  L_2 & R_2 \\
  L_1 & (10, 9) & (0, 0) \\
  R_1 & (0, 0) & (9, 10)
\end{pmatrix}
\]

  - For “Reporting a Crime” there is a symmetry for each permutation of the players.
For a definition like the following, it’s best to “know” what it says before you read it.

Symmetry

A symmetry of a game \((S_1, \ldots, S_n; u_1, \ldots, u_n)\) consists of a permutation (one-to-one onto function)

\[
\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}
\]

along with an \(n\) tuple of one-to-one functions

\[
\begin{align*}
r_1 : S_1 &\to S_{\sigma(1)} \\
&
\vdots \\
r_n : S_n &\to S_{\sigma(n)}
\end{align*}
\]

such that

\[
u_{\sigma(i)}(r(s)) = u_i(s)
\]

for all \(i\) and all \(s \in S\), where we define \(r : S \to S\) by setting

\[
r(s) = (r_1(s_1), \ldots, r_n(s_n)).
\]
Once you have one pure equilibrium of a symmetric game, one can apply all available symmetries to generate others.

**Theorem:** Given a symmetry as above, if \( s \) is a pure Nash equilibrium, then so is \( r(s) \).

**Proof.** Suppose not. Then there is some \( j \) and \( t_j \in S_j \) such that \( u_j(t_j, r(s)_{-j}) > u_j(r(s)) \). Let \( i = \sigma^{-1}(j) \). Then

\[
\begin{align*}
  u_i(r_i^{-1}(t_j), s_{-i}) &= u_j(r(r_i^{-1}(t_j), s_{-i})) \\
  &= u_j(t_j, r(s)_{-j}) \\
  &> u_j(r(s)) \\
  &= u_i(s),
\end{align*}
\]

contradicting the assumption that \( s \) is a Nash equilibrium.
This also works for mixed strategies:

- Define \( r_i : \Delta(S_i) \rightarrow \Delta(S_{\sigma(i)}) \) by setting
  \[
  r_i(\alpha)(r_i(s_i)) = \alpha_i(s_i).
  \]
  - That is, the probability that \( r_i(\alpha_i) \) assigns to \( r_i(s_i) \) is the same as the probability that \( \alpha_i \) assigns to \( s_i \).

- For a mixed strategy profile \( \alpha \) let
  \[
  r(\alpha) = (r_1(\alpha_1), \ldots, r_n(\alpha_n)).
  \]

- Applying the distributive law to the definition of expected utility gives the following formula:
  \[
  u_{\sigma(i)}(r(\alpha)) = u_i(\alpha)
  \]
  for all \( i \) and all mixed strategy profiles \( \alpha \).

**Theorem:** Given a symmetry as above, if \( \alpha \) is a mixed Nash equilibrium, then so is \( r(\alpha) \).

**Proof.** Just the same as above, except now with mixed strategies.
Existence

The most important theorem in game theory is the Nash equilibrium existence theorem:

**Theorem:** If there are finitely many players and each player has finitely many pure strategies, then there is at least one Nash equilibrium.

**Proof Idea:** Apply a generalization of the Brouwer fixed point theorem to the correspondence

\[ \alpha \mapsto BR_1(\alpha_{-1}) \times \cdots \times BR_n(\alpha_{-n}). \]

- This is the main result in John Nash’s thesis.
  - Relative to the state of knowledge at the time, it is actually rather trivial.
  - Nash went on to prove several extremely deep and important results before developing schizophrenia.
Brouwer’s Fixed Point Theorem: Let $D^m$ be the $m$-dimensional disk consisting of those points in $\mathbb{R}^m$ whose distance from the origin is less than or equal to one. If $f : D^m \to D^m$ is continuous, then $f$ has a fixed point, i.e., a point $x^* \in D^m$ such that

$$f(x^*) = x^*.$$ 

- This was first proved around 1910. It is one of the most famous and important theorems in the field of mathematics called topology.

- You will probably not be able to understand very much of it, but even so you might learn a bit about various proofs of the BFT by looking at “From Imitation Games to Kakutani” (joint with Rabee Tourky) on my research page.
Symmetric Equilibrium

- A mixed strategy is *symmetric* if \( r(\alpha) = \alpha \) for every symmetry \((\sigma, r)\).

- A *symmetric Nash equilibrium* is a Nash equilibrium that is a symmetric mixed strategy.

Nash pointed out that (with a little fiddling) his argument actually proves:

**Theorem:** *There is at least one symmetric Nash equilibrium.*

- Ideas based on symmetry:
  - Starting with one equilibrium, use symmetries to generate others.
  - Divide the problem into two parts:
    * finding symmetric equilibria;
    * finding asymmetric equilibria.
  - If two symmetric players are doing different things, it is “unlikely” that both are happy to mix.
Ideas Related to Dominance

Strict Dominance:

- If there is a strictly dominated strategy, the Nash equilibria are the same as the Nash equilibria of the truncated game resulting from its elimination.

- When solving for all the Nash equilibria, before doing anything else eliminate strictly dominated equilibria repeatedly until there are none left in the truncated game.

- Example:

  \[
  \begin{array}{ccc}
  L & M & R \\
  U & (2, 3) & (1, 2) & (6, 0) \\
  C & (1, 2) & (7, 1) & (5, 0) \\
  D & (0, 0) & (6, 5) & (6, 4) \\
  \end{array}
  \]
Weak Dominance:

- If there is a strictly dominated strategy, then:
  - the Nash equilibria of the truncated game resulting from its elimination are all equilibria of the entire game;
  - in any equilibrium in which the weakly dominated strategy has positive probability, the complementary strategy vectors for which it is strictly worse than some dominating strategy have probability zero.

- Example:

\[
\begin{array}{ccc}
L & M & R \\
U & (2,3) & (1,0) & (0,1) \\
D & (2,2) & (0,1) & (1,0) \\
\end{array}
\]
Example: Rock-Scissors-Paper

- Problem Description:
  - The two players simultaneously each choose Rock, Scissors, or Paper. Rock smashes Scissors, Scissors cuts Paper, and Paper wraps Rock.

\[
\begin{bmatrix}
R_2 & S_2 & P_2 \\
R_1 & (0,0) & (1,-1) & (-1,1) \\
S_1 & (-1,1) & (0,0) & (1,-1) \\
P_1 & (1,-1) & (-1,1) & (0,0)
\end{bmatrix}
\]

- There are lots of symmetries:
  (a) We can rotate the strategies:

\[
R_1 \rightarrow S_1, \quad S_1 \rightarrow P_1, \quad P_1 \rightarrow R_1, \\
R_2 \rightarrow S_2, \quad S_2 \rightarrow P_2, \quad P_2 \rightarrow R_2.
\]

  (b) We can also interchange the players:

\[
R_1 \rightarrow R_2, \quad R_2 \rightarrow R_1, \\
S_1 \rightarrow S_2, \quad S_2 \rightarrow S_1, \\
P_1 \rightarrow P_2, \quad P_2 \rightarrow P_1.
\]
• The only symmetric strategy profile is for both players to assign probability $1/3$ to each pure strategy. Denote this by $\alpha^*$. Since a symmetric Nash equilibrium exists, without doing any calculations we know that $\alpha^*$ is a Nash equilibrium.

• In the truncated game resulting from the elimination of $R_1$, $P_2$ is strictly dominated by $S_2$.
  
  – In view of the symmetries, it follows that once we eliminate one pure strategy, iterative eliminations of strictly dominated strategies eliminate all the others.
  
  – Therefore there is no Nash equilibrium that is not totally mixed.

• We could do a calculation to show that $\alpha^*$ is the only totally mixed Nash equilibrium, but it is also interesting to give a “calculation free” proof by contradiction.
• Suppose that \( \alpha^{**} \) is another (necessarily totally mixed) Nash equilibrium.
  
  - Without loss of generality (by symmetry!) suppose that \( \alpha_{1}^{**} \neq \alpha_{1}^{*} \).
  
  - For any real number \( t \) such that
    
    \[ (1 - t)\alpha_{1}^{*} + t\alpha_{1}^{**} \]
    
    is a mixed strategy (i.e., all probabilities are nonnegative)
    
    \[ ((1 - t)\alpha_{1}^{*} + t\alpha_{1}^{**}, \alpha_{2}^{*}) \]
    
    is a Nash equilibrium:
    
    * since \( \alpha_{1}^{*} \) and \( \alpha_{1}^{**} \) are both best responses to \( \alpha_{2}^{*} \), so is \( (1 - t)\alpha_{1}^{*} + t\alpha_{1}^{**} \);
    
    * since \( \alpha_{2}^{*} \) is a best response to both \( \alpha_{1}^{*} \) and \( \alpha_{1}^{**} \), it is a best response to
      
      \[ (1 - t)\alpha_{1}^{*} + t\alpha_{1}^{**} \].
    
  - There is some \( t \) such that all components of \( (1 - t)\alpha_{1}^{*} + t\alpha_{1}^{**} \) are nonnegative and at least one is zero, giving a Nash equilibrium that is not totally mixed, which we know is impossible.