Economics 3012
Strategic Behavior
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Lecture 3

Topics

• Problem Set 2
• Hotelling’s Model Of Competitive Elections
• The War of Attrition
• Accident Law
• Introduction to Auction Theory
Problem Set 2

Exercise 49.1

Problem Description:

- Each voter votes for A, B, or C. A tie between two candidates is regarded as worse than the better one winning and better than the worse one winning.
  - Show that voting for one’s least favorite candidate is the only weakly dominated vote.
  - Find a Nash equilibrium in which some voter votes for her second favorite candidate.
Analysis:

- Voting for the least favorite candidate is weakly dominated.
  - Specifically, if a voter switches from her least favorite to her favorite, that can only increase the chance that her favorite candidate wins and diminish the chance that her least favorite wins.

- Voting for one’s favorite candidate may easily be the unique best response.
  - It may result in that candidate’s election, so this strategy cannot be weakly dominated.

- Voting for one’s second favorite candidate cannot be weakly dominated.
  - This is illustrated in the example below.
• Example:
  – There are five voters.
    * Voters 1 and 2 prefer $A$ to $B$ and $B$ to $C$.
    * Voter 3 prefers $B$ to $A$ and $A$ to $C$.
    * Voters 4 and 5 prefer $C$ to $B$ and $B$ to $A$.
  – Voters 1, 2, and 3 vote for $A$ and Voters 4 and 5 vote for $C$.
  – Voting for $A$ is the unique best response for Voter 3, even though $B$ is her favorite candidate.
  – This is a Nash equilibrium.
    * If any of the first three voters switch to a different vote, the outcome changes to either $C$ or a tie between $A$ and $C$, which is worse than $A$ for each of them.
    * Voters 4 and 5 cannot change the outcome.
Exercise 61.1

Problem Description:

• Each firm \( i = 1, \ldots, n \) chooses \( q_i \).

• Price is \( P(Q) = \alpha - Q \) where
  \[ Q = q_1 + \cdots + q_n. \]

• Firm \( i \)'s profit is
  \[ \pi_i(q_1, \ldots , q_n) = q_i(P(Q) - c). \]

Analysis

• If \( \alpha - \sum_{j \neq i} q_j < c \), then the unique best response is \( q_i = 0 \).

• If \( \alpha - \sum_{j \neq i} q_j \geq c \), then setting the partial derivative of \( \pi_i \) with respect to \( q_i \) equal to zero shows that the unique best response is
  \[ q_i^*(Q) = \frac{1}{2} \left( \alpha - \sum_{j \neq i} q_j - c \right). \]
• \( \alpha - \sum_{j \neq i} q_j < c \) is impossible because each \( i \) would do better setting \( q_i = 0 \).

• No agent will set \( q_i = 0 \).
  - This would only happen if
    \[ \alpha - \sum_{j \neq i} q_j = c. \]
  - But then for any \( h \) with \( q_h > 0 \) we have
    \[ \alpha - \sum_{j \neq h} q_j > c \]
    and
    \[
    \sum_{j \neq i} q_j = \sum_i q_i = q_h + \sum_{j \neq h} q_j
    < (\alpha - \sum_{j \neq h} q_j - c) + \sum_{j \neq h} q_j = \alpha - c.
    \]

• For any two \( i \) and \( h \) we have \( q_i = q_h \) because
    \[
    q_i - q_h = \frac{1}{2} (\alpha - \sum_{j \neq i} q_j - c) - \frac{1}{2} (\alpha - \sum_{j \neq h} q_j - c)
    = \frac{1}{2} (q_i - q_h).
    \]
    Thus all firms produce a common quantity \( q \).
• Therefore we have

\[ q = \frac{1}{2}(\alpha - (n - 1)q - c), \]

so \( q = (\alpha - c)/(n + 1). \)

• The resulting price is

\[ \alpha - n(\alpha - c)/(n + 1) = \frac{1}{n+1}\alpha + \frac{n}{n+1}c \]

which clearly approaches \( c \) as \( n \to \infty \).
Downsian Elections

Hotelling’s model was popularized in political science by a 1957 book *An Economic Theory of Democracy* by Downs. In the Downsian application of Hotelling’s model it is more natural to assume that the goal is to capture the largest number of voters.

Problem Description:

- Two political parties $i = 1, 2$ each choose policy positions $x_i$ with $0 \leq x_i \leq 1$.
- Voters have *single peaked preferences*: each has a “bliss point,” and in each direction likes policy positions less as they go further from this.
- To simplify the mathematical description we assume that departures from the bliss point in the two directions are symmetric, so the voter votes for the party whose position is closer.
  - This assumption is unnecessary.
• We also assume that voters’ bliss points are uniformly distributed across locations between 0 and 1.

– This assumption is also unnecessary.

• Each political party wins if it gets a majority of votes. Let

\[ v_i(x_1, x_2) = \begin{cases} 
\frac{1}{2}(x_i + x_j), & x_i < x_j, \\
1/2, & x_i = x_j, \\
1 - \frac{1}{2}(x_i + x_j), & x_i > x_j.
\end{cases} \]

Then

\[ \pi_i(x_1, x_2) = \begin{cases} 
1, & v_i(x_1, x_2) > v_j(x_1, x_2), \\
1/2, & v_i(x_1, x_2) = v_j(x_1, x_2), \\
0, & v_i(x_1, x_2) > v_j(x_1, x_2).
\end{cases} \]
Analysis:

- The best response correspondence for agent $i$ is

$$B_i(x_j) = \begin{cases} (x_j, 1 - x_j), & x_j < 1/2, \\ \{1/2\}, & x_j = 1/2, \\ (1 - x_j, x_j), & x_j > 1/2. \end{cases}$$

- The only Nash equilibrium is

$$(x_1, x_2) = (1/2, 1/2).$$
Exercise 74.2

Problem Description:

The winner of the bigger state will have a majority in the electoral college and win the election.

Analysis

- If the other candidate is not choosing $m_1$, you can win all the electoral votes of State 1, and the election, by choosing a position closer to $m_1$.

- Therefore all Nash equilibria have at least one agent choosing $m_1$.

- If the other agent is choosing $m_1$ your unique best response is also to choose $m_1$.

- Thus $(m_1, m_1)$ is a Nash equilibrium, and there is no other.
Multidimensional Issue Spaces

• The single dimensional Downsian model suggests that politics will be stable, with a centrist tendency.

• But it may happen that there is more than one dimension of policy that voters care about. For example:
  – Progressivity of taxation.
  – Treatment of minorities.

• In the multidimensional case the Downsian model seems highly unstable: there is typically no equilibrium.
Problem (not from Osborne) Description:

- There are three voters $v = 1, 2, 3$:
  - Each voter $v$ has a bliss point $b_v \in \mathbb{R}^2$.
  - For any two policy positions $p_1, p_2 \in \mathbb{R}^2$, voter $v$ prefers whichever is closer to $b_v$.
    * The voter’s indifference curves are circles centered at $b_v$.
  - Assume $b_1$, $b_2$, and $b_3$ are not colinear. (‘Colinear’ means “contained in a single line.”)

- Two political parties choose policy positions $p_1, p_2 \in \mathbb{R}^2$.

- A party wins if a majority of the voters prefer its position.
Analysis:

- Assuming that $p_1 \neq p_2$, let $L$ be the line that
  - is perpendicular to the line segment between $p_1$ and $p_2$;
  - contains the midpoint of the line segment between $p_1$ and $p_2$.

- Then $L$ divides the voters who vote for party 1 from those who vote for party 2.

- For a given $p_1$ and any line $\ell$ that does not contain $p_1$, one can choose $p_2$ to make $L = \ell$.

- Since $b_1$, $b_2$, and $b_3$ are not colinear, $p_1$ is not in the line $\ell^*$ containing some pair of bliss points, say $b_1$ and $b_2$.

- The second party can win by choosing $p_2$ to make $L$ a line parallel to $\ell^*$ with $p_1$ on one side and $b_1$ and $b_2$ on the other.
  - Thus there is no Nash equilibrium.
The War of Attrition

Problem Description:

- Two contestants are disputing some item of value.
- The one who concedes first loses. Ties are decided by coin flip.
- Waiting time is costly for both, and both incur the cost up to the time of concession.

Let $v_1$ and $v_2$ be the values of the object to the two players, and let $t_1$ and $t_2$ be the concession times.

- Agent $i$’s payoff is

\[
 u_i(t_1, t_2) = \begin{cases} 
-t_i, & t_i < t_j, \\
\frac{1}{2}v_i - t_i, & t_i = t_j, \\
v_i - t_j, & t_i > t_j.
\end{cases}
\]
Analysis:

• Agent $i$’s best response correspondence is

$$B_i(t_j) = \begin{cases} 
\{ t_i : t_i > t_j \}, & t_j < v_i, \\
\{0\} \cup \{ t_i : t_i > t_j \}, & t_j = v_i, \\
\{0\}, & t_j > v_i.
\end{cases}$$

• There cannot be a Nash equilibrium $(t_1, t_2)$ with $t_1, t_2 > 0$.

• Therefore the set of Nash equilibria is

$$\{ (0, t_2) : t_2 \geq v_1 \} \cup \{ (t_1, 0) : t_1 \geq v_2 \}.$$

• Note that:
  – There is never a fight in equilibrium.
  – Agent $i$ can win even if $v_i < v_j$.
  – The only equilibria are asymmetric.

• If you expect to play a War of Attrition later, how might you prepare for this?

• If Wars of Attrition are frequent, a reputation for stubbornness can be advantageous.
**Variant: Waiting Reduces Values**

**Problem Description:**

- The object has initial values $v_1$ and $v_2$.
- Each player $i = 1, 2$ chooses $t_i \geq 0$.
- Agent $i$’s payoff is

$$
u_i(t_1, t_2) = \begin{cases} 
0, & t_i < t_j, \\
\frac{1}{2}(v_i - t_i), & t_i = t_j < v_i, \\
0, & t_i = t_j \geq v_i, \\
v_i - t_j, & t_i > t_j < v_i, \\
0, & t_i > t_j \geq v_i.
\end{cases}
$$

**Analysis:**

- Let $(t_1, t_2)$ be a Nash equilibrium.
  - Both $t_1 < v_2$ and $t_2 < v_1$ is impossible.
  - If $t_i \geq v_j$, then for any $t_j$, $(t_1, t_2)$ is a Nash equilibrium.

- Therefore the set of Nash equilibria is

$$\{(t_1, t_2) : t_1 \geq v_2 \text{ or } t_2 \geq v_1 \}.$$
Accident Law

A rule of law defines the rules of a game.

• How will the game be played?
• Which rules lead to efficient (or at least better) outcomes?
• This is the spirit of mechanism design.
  – There is some inability of governing institutions to directly dictate desired outcomes.
    ∗ This may arise due to inadequate tools of enforcement or private information.
  – One hopes to design a system that (in the case of privately held information) work reasonably well in a broad range of possibilities.
Problem Description:

- Agent 1 (the *injurer*) and agent 2 (the *victim*) choose levels of care $a_1$ and $a_2$ that are nonnegative real numbers.

- The loss (on average) resulting from these choices is $L(a_1, a_2)$. Assume that the function $L$ is:
  - positive for all $(a_1, a_2)$;
  - decreasing in each variable:
    $$L(a_1, a_2) > L(a'_1, a_2) \text{ and } L(a_1, a_2) > L(a_1, a'_2) \text{ whenever } a'_1 > a_1 \text{ and } a'_2 > a_2.$$

- A rule of law is a function $\rho(a_1, a_2) \in [0, 1]$ assigning a share of the loss to the injurer.
  - This results in a game in which the choice variables are the two levels of care and the payoff functions are:
    $$u_1(a_1, a_2) = -a_1 - \rho(a_1, a_2)L(a_1, a_2);$$
    $$u_1(a_1, a_2) = -a_2 - (1-\rho(a_1, a_2))L(a_1, a_2).$$
• Negligence with contributory negligence: there are mandated levels of care \( X_1 \) and \( X_2 \) for the two agents such that:

\[
\rho(a_1, a_2) = \begin{cases} 
1, & a_1 < X_1 \text{ and } a_2 \geq X_2, \\
0, & \text{otherwise.}
\end{cases}
\]

• Special cases:
  - Pure negligence: \( X_1 > 0 \) and \( X_2 = 0 \).
  - Strict liability: \( X_1 = \infty \) and \( X_2 = 0 \).
  - Efficient rule: \( (X_1, X_2) = (\hat{a}_1, \hat{a}_2) \) where \( (\hat{a}_1, \hat{a}_2) \) maximizes the total loss: for all \( a_1, a_2 \)

\[
-\hat{a}_1 - \hat{a}_2 - L(\hat{a}_1, \hat{a}_2) \geq -a_1 - a_2 - L(a_1, a_2).
\]
Analysis

- If \((X_1, X_2) = (\hat{a}_1, \hat{a}_2)\), then \((\hat{a}_1, \hat{a}_2)\) is a Nash equilibrium.
  - The injurer gains nothing by increasing the level of care, and for any \(a_1 < \hat{a}_1\) we have
    
    \[-a_1 - L(a_1, \hat{a}_2) < -\hat{a}_1 - L(\hat{a}_1, \hat{a}_2) \leq \hat{a}_1.\]
  - The victim’s payoff \(-a_2 - L(\hat{a}_1, a_2)\) is maximized by setting \(a_2 = \hat{a}_2\).
• $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium.
  
  – Let $(a_1, a_2)$ be a Nash equilibrium.
  
  – The injurer’s best response correspondence satisfies:

$B_1(a_2) = \begin{cases} 
\emptyset, & a_2 < \hat{a}_2, \\
\{\hat{a}_1\}, & a_2 = \hat{a}_2, \\
\subset [0, \hat{a}_1], & a_2 > \hat{a}_2.
\end{cases}$

– If $a_1 = 0$, the victim does better choosing $a_2 = \hat{a}_2$ than any $a_2 < \hat{a}_2$ because

$$-\hat{a}_2 \geq -\hat{a}_1 - \hat{a}_2 - L(\hat{a}_1, \hat{a}_2)$$

$$> -0 - a_2 - L(0, a_2).$$

* Therefore $a_2 \geq \hat{a}_2$ and $a_1 \leq \hat{a}_1$.

– If $a_1 < \hat{a}_1$, then for the victim $\hat{a}_2$ is better than any $a_2 > \hat{a}_2$ because the respective payoffs are $-\hat{a}_2$ and $-a_2$.

* Therefore $a_2 = \hat{a}_2$, and $a_1$ must be $\hat{a}_1$. 
Introduction to Auctions

Auctions are very old.

Auctions are increasingly important.

- Government procurement.
- Privatization, including spectrum.
- Internet auctions.
- Fish, flowers, wine, art, etc.

Types of auctions:

- First price.
  - First price sealed bid.
  - Dutch.

- Second price.
  - Second price sealed bid.
  - English or open outcry.
Second Price Sealed Bid Auctions

Problem Description:

• Each player $i = 1, \ldots, n$ has a value $v_i$.
• Each player submits a bid $b_i$.
• The object is awarded to the player submitting the highest bid, who pays the second highest bid:

$$u_i(b_1, \ldots, b_n) = v_i - \max\{ b_j : j \neq i \}$$

if $b_i = \max_j b_j$, and $u_i(b_1, \ldots, b_n) = 0$ otherwise. (Assume the winner is chosen by equiprobable lottery when there is a tie.)
Analysis:

- Bidding your value is a weakly dominant strategy:
  - Your bid affects only whether you win, and not how much you pay when you win.
  - $b_i < v_i$ rather than $b_i = v_i$ risks losing when winning would be profitable.
  - $b_i > v_i$ rather than $b_i = v_i$ risks paying too much.

- There are many equilibria.
First Price Sealed Bid Auctions

Problem Description:

- Each player $i = 1, \ldots, n$ has a value $v_i$.
- Each player submits a bid $b_i$.
- The object is awarded to the player submitting the highest bid, and that person pays their bid:

$$u_i(b_1, \ldots, b_n) = v_i - b_i$$

if $b_i = \max_j b_j$, and $u_i(b_1, \ldots, b_n) = 0$ otherwise. (Assume the winner is chosen by equiprobable lottery when there is a tie.)

Analysis: if the values $v_1 > \cdots > v_n$ are known by everyone, in any Nash equilibrium $b_1 = v_2$ and bidder one wins the object with probability one.
Multiunit Auctions

Case I: Identical objects: suppose the $n$ bidders are competing for $m$ identical objects. Each bidder submits a list $b_i = (b^1_i, \ldots, b^k_i)$ with $b^1_i \geq \cdots \geq b^k_i$. The objects are awarded to the $m$ largest bids.

- There are several payment rules that can be used:
  - Discriminatory: each winning bid pays the amount bid.
  - Uniform price: all winning bids pay the highest rejected bid.
  - Vickrey: A bidder winning $k$ objects pays the sum of the $k$ highest bids submitted by other bidders.
Two Object Case

Problem Description:

- There are two identical objects being sold.
- For each player the value of two objects is less than twice the value of a single object.
- Each player $i = 1, \ldots, n$ submits bids $v_i > w_i > 0$.
- For which of the payment rules above is “truth telling” (bidding your actual valuations) a weakly dominant strategy?

Analysis:

- In the Vickrey auction your bid affects whether you win but not how much you pay when you win.
  - Truth telling is the only rule that insures that you win if and only if it is advantageous.
- Examples show that the other two rules do not have this property.
Case II: Nonidentical Objects and Positive Complementarities

- Leaving incentives aside, hard problems abound:
  - Suppose each \( i = 1, \ldots, n \) has a value \( v_{ik} \) for each object \( k = 1, \ldots, m \).
    * How hard is it (in a computational sense) to find an allocation of at most one object to each bidder that maximizes the sum of values?
  - When bidders’ values of packages of objects may differ from the sum of values of individual objects, the value maximization problem is computationally intractable.

- Spectrum Auctions in the US
  - Simultaneous ascending bid.
    * The intent was to encourage information flows.
  - It was vulnerable to implicit collusion.
  - Results in practice were variable.
Chopstick Auction

Problem (not in Osborne) Description:

• Each $i = 1, 2$ submits bids $(b^1_i, b^2_i, b^3_i)$ on three identical objects.

• Each object is sold to the high bidder, with the bidder paying her bid.
  – In the event of a tie, the winner is decided by coin flip.

• Each agent losses the money spent on winning bids and wins $1$ if she wins two or more objects.

• That is, one chopstick is worth nothing, and the value of three chopsticks is the same as the value of two.
Analysis:

- There is no Nash equilibrium.
  - Say that player $i$’s strategy is vulnerable if $b_i^k + b_i^\ell < 1$ for some distinct $k, \ell = 1, 2, 3$.
  - In any Nash equilibrium one player’s strategy must be invulnerable, and the other player must win no objects.
  - If, in equilibrium, player $i$ is playing the invulnerable strategy $(b_i^1, b_i^2, b_i^3)$, then player $i$ wins all three objects, paying
    \[
    b_i^1 + b_i^2 + b_i^3 = \frac{1}{2}(b_i^1 + b_i^2) + \frac{1}{2}(b_i^1 + b_i^3) + \frac{1}{2}(b_i^2 + b_i^3) \geq \frac{3}{2}
    \]
    and could do better by bidding $(0, 0, 0)$.
- A remarkable, and remarkably difficult, paper by Rosenthal and Szentes gives an equilibrium in mixed strategies.